Lecture 7: State Feedback Control System

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SIST 1D#206

Observer Design for Better Disturbance Rejection

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- Example paper using frequency responses to evaluate a servo control system

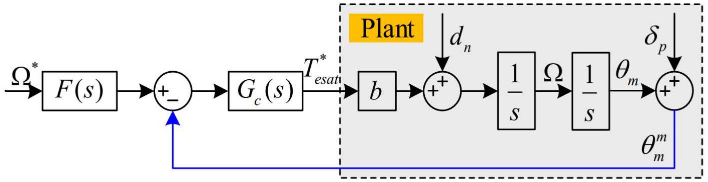

Fig. 3. Block diagram of different ADRC systems in transfer function form

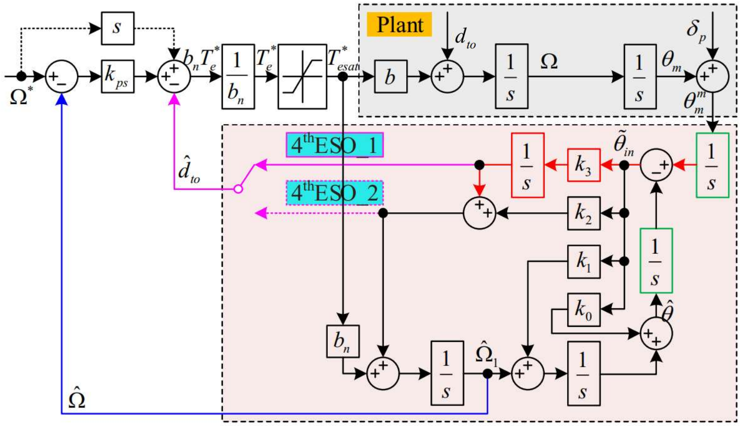

Fig. 14. The equivalent block diagram of the system 4^{th}ESO_1-2.

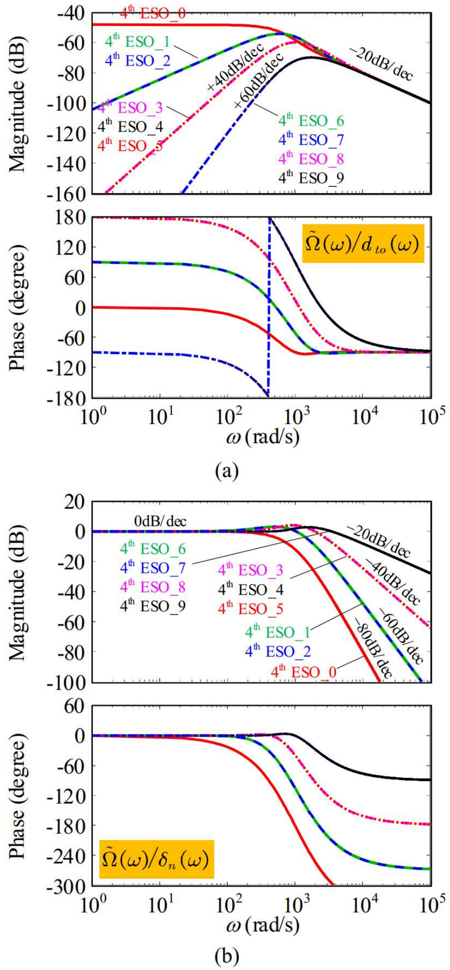

Fig. 10. Bode diagrams of the ten fourth order ESOs for observing speed.

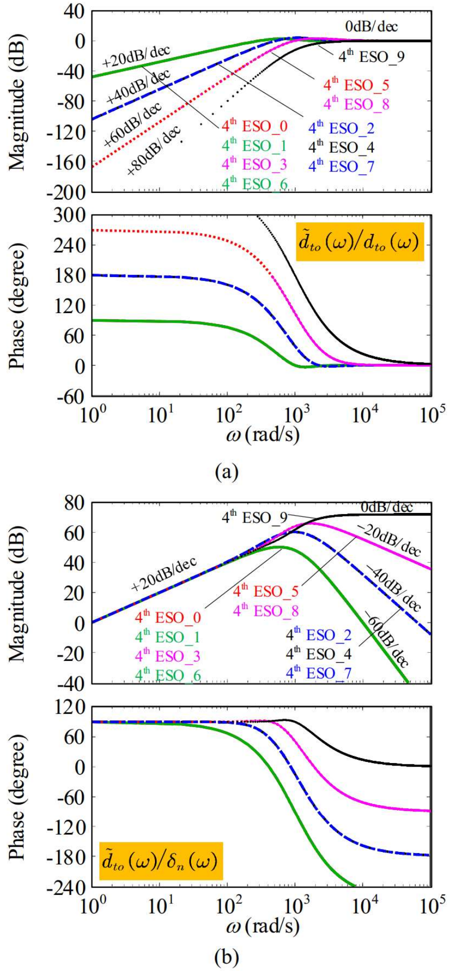

Fig. 11. Bode diagrams of the ten fourth order ESOs for observing disturbance.

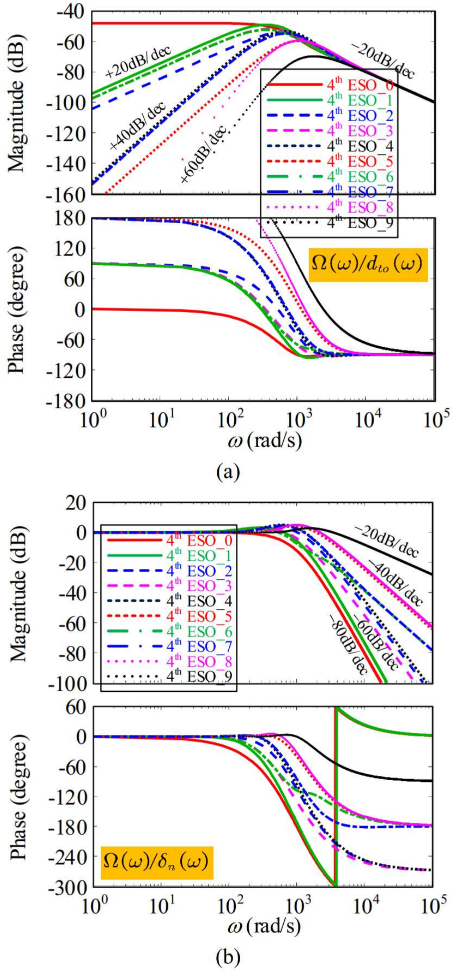

Fig. 12. Bode diagram of the ten systems under the same p_o.

Zuo et al., "Different Active Disturbance Rejection Controllers Based on the Same Order GPI Observer", TIE, 2021

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State Space Model (Review)

State space model

- The simultaneous **first-order differential** equations about the state variables
  - State variables must be linearly independent; that is, no state variable can be written as a linear combination of the other state variables.
- **Algebraic** output equation

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}$$

$$\boxed{\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}}$$

$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

- E.g.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & K_T J^{-1}\\ 0 & -K_E L^{-1} & -RL^{-1}\end{bmatrix}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix} + \begin{bmatrix}0\\ 0\\ L^{-1}\end{bmatrix}u + \begin{bmatrix}0\\ J^{-1}\\ 0\end{bmatrix}\tau_L$$

$$y = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix}$$

- State space: The n-dimensional space whose axes are the state variables.

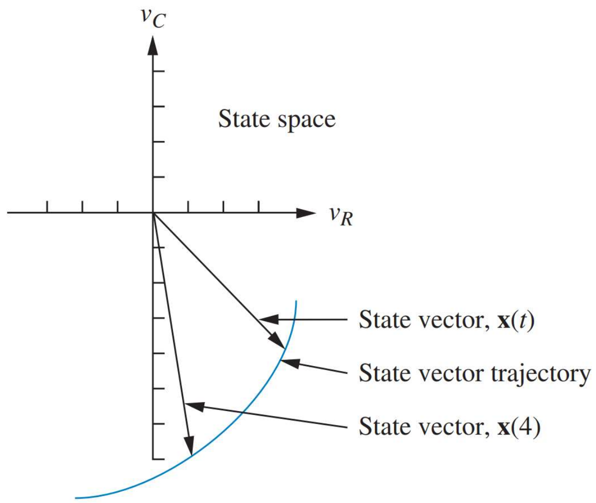

FIGURE 3.3 Graphic representation of state space and a state vector

PROBLEM: Find the state-space representation of the electrical network shown in Figure 3.8. The output is $v_o(t)$.

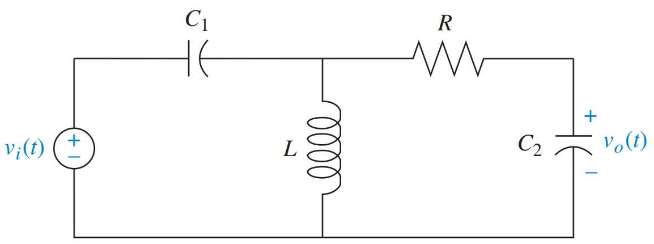

FIGURE 3.8 Electric circuit for Skill-Assessment Exercise 3.1

ANSWER:

PROBLEM: Represent the translational mechanical system shown in Figure 3.9 in state space, where $x_3(t)$ is the output.

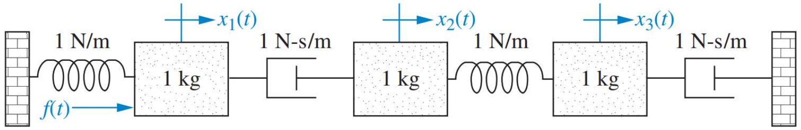

FIGURE 3.9 Translational mechanical system for Skill-Assessment Exercise 3.2

ANSWER:

From O.D.E to State space model

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At first we select a set of state variables, called *phase variables*, where each subsequent state variable is defined to be the derivative of the previous state variable.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

$$x_1 = y,\quad x_2 = \frac{dy}{dt},\quad x_3 = \frac{d^2y}{dt^2},\quad \vdots \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

$$\dot{x}_1 = \frac{dy}{dt},\quad \dot{x}_2 = \frac{d^2y}{dt^2},\quad \dot{x}_3 = \frac{d^3y}{dt^3},\quad \vdots \quad \dot{x}_n = \frac{d^ny}{dt^n}$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_{n-1} &= x_n\\ \dot{x}_n &= -a_0x_1 - a_1x_2 \cdots - a_{n-1}x_n + b_0u\end{aligned}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}0&1&0&0&0&0&\cdots&0\\ 0&0&1&0&0&0&\cdots&0\\ 0&0&0&1&0&0&\cdots&0\\ \vdots\\ 0&0&0&0&0&0&\cdots&1\\ -a_0&-a_1&-a_2&-a_3&-a_4&-a_5&\cdots&-a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + \begin{bmatrix}0\\0\\0\\ \vdots\\ 0\\ b_0\end{bmatrix}u \tag{3.52}$$

$$y = \begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}$$

From transfer function to state space model (1)

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Converting a Transfer Function with a Constant Term in the Numerator

$$\frac{C(s)}{R(s)}=\frac{24}{(s^3+9s^2+26s+24)}$$

$$(s^3+9s^2+26s+24)C(s)=24R(s)$$

$$\dddot{c}+9\ddot{c}+26\dot{c}+24c=24r$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\ 0&0&1\\ -24&-26&-9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 24\end{bmatrix}r$$

$$y=\begin{bmatrix}1&0&0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

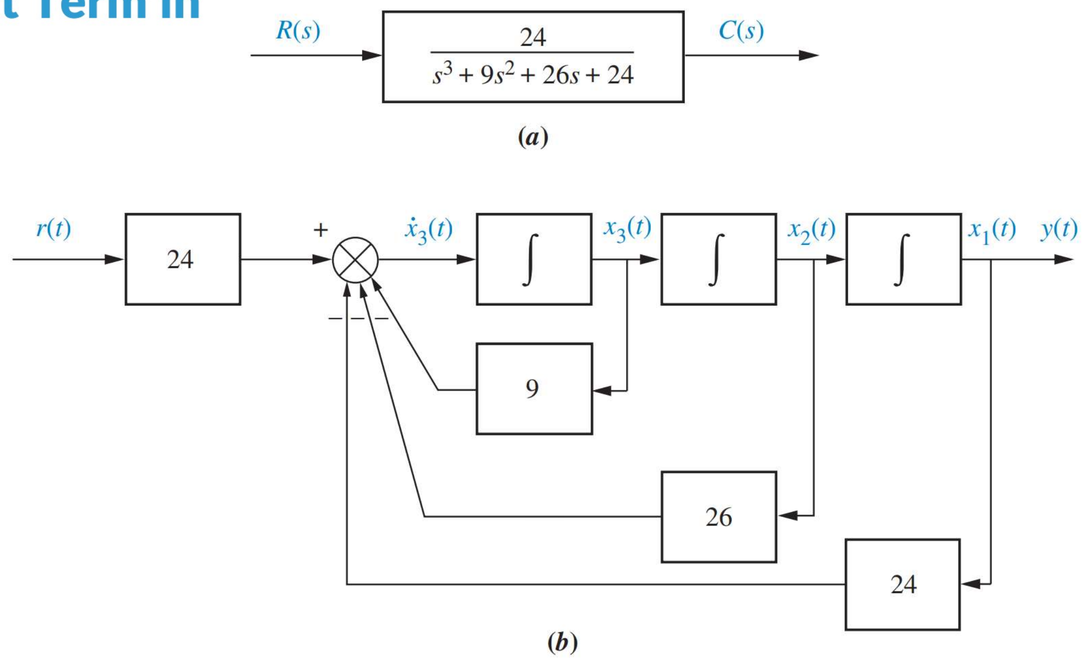

FIGURE 3.10 **a.** Transfer function; **b.** equivalent block diagram showing phase variables. Note: $y(t) = c(t)$.

From transfer function to state space model (2)

Converting a Transfer Function with a Polynomial in the Numerator

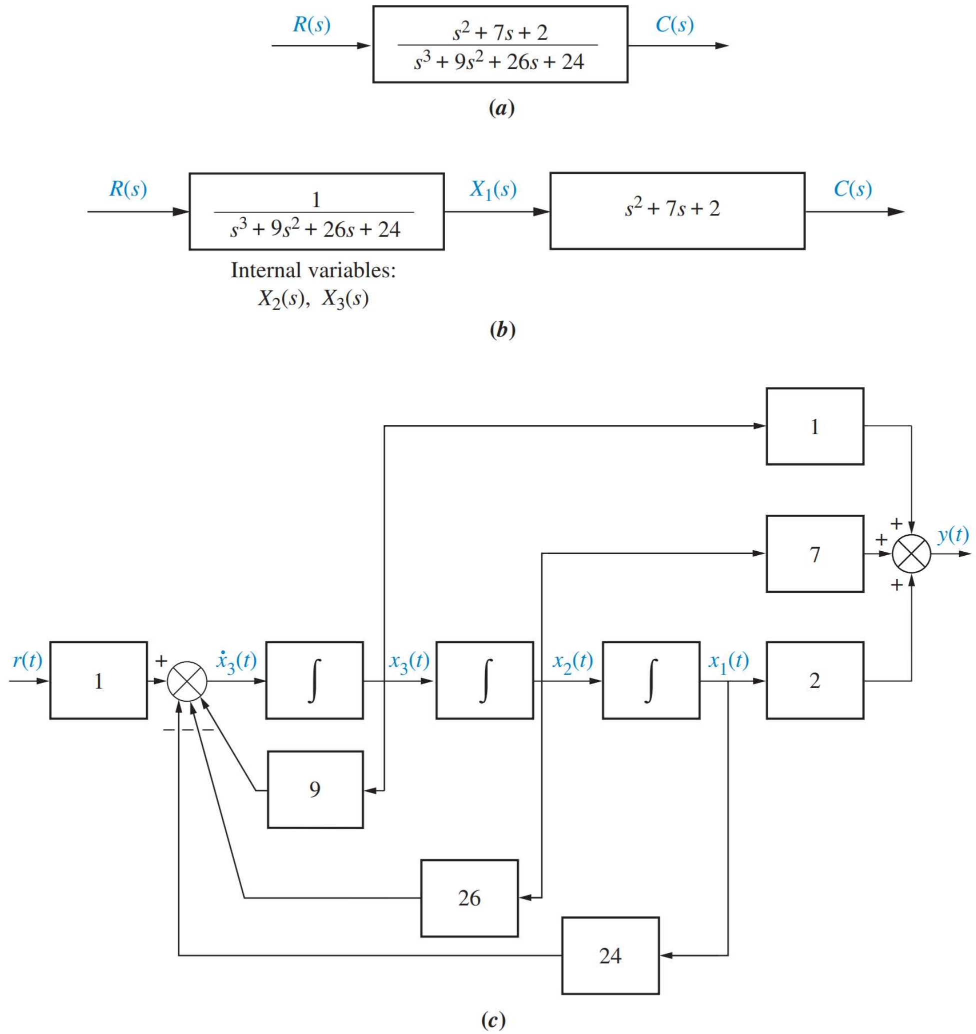

FIGURE 3.12 **a.** Transfer function; **b.** decomposed transfer function; **c.** equivalent block diagram Note: $y(t) = c(t)$.

From transfer function to state space model (2)

PROBLEM: Find the state equations and output equation for the phase-variable representation of the transfer function $G(s) = \dfrac{2s+1}{s^2+7s+9}$.

ANSWER:

State space model to transfer funtion (ss2tf)

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

Transfer function matrix

Single input single output

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

State space model to transfer funtion (ss2tf)

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PROBLEM: Given the system defined by Eq. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u \qquad (3.74a)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x} \qquad (3.74b)$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

https://en.wikipedia.org/wiki/Adjugate_matrix#:~:text=In%20linear%20algebra%2C%20the%20adjugate,matrix%20is%20the%20conjugate%20transpose.

State space model to transfer funtion (ss2tf)

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TryIt 3.2

Use the following MATLAB and the Control System Toolbox statements to obtain the transfer function shown in Skill-Assessment Exercise 3.4 from the state-space representation of Eq. (3.78).

```
A=[-4 -1.5; 4 0];
B=[2 0]';
C=[1.5 0.625];
D=0;
T=ss(A, B, C, D);
T=tf(T)
```

PROBLEM: Convert the state and output equations shown in Eq. (3.78) to a transfer function.

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t) \tag{3.78a}$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}\mathbf{x} \tag{3.78b}$$

ANSWER:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \operatorname{adj}(\mathbf{A}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

State space model is not unique

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu}$$

$$\mathbf{y} = \mathbf{CPz} + \mathbf{Du}$$

PROBLEM: Given the system represented in state space by Eqs. (5.73),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \tag{5.73a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x} \tag{5.73b}$$

transform the system to a new set of state variables, $\mathbf{z}$, where the new state variables are related to the original state variables, $\mathbf{x}$, as follows:

$$z_1 = 2x_1 \tag{5.74a}$$

$$z_2 = 3x_1 + 2x_2 \tag{5.74b}$$

$$z_3 = x_1 + 4x_2 + 5x_3 \tag{5.74c}$$

$$\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\mathbf{x} = \mathbf{P}^{-1}\mathbf{x}$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

Diagonal state space representation

Eigenvector. The eigenvectors of the matrix $\mathbf{A}$ are all vectors, $\mathbf{x_i} \neq \mathbf{0}$, which under the transformation $\mathbf{A}$ become multiples of themselves; that is,

$$\boxed{\mathbf{A}\mathbf{x_i} = \lambda_i \mathbf{x_i}} \tag{5.80}$$

where λ_i's are constants.

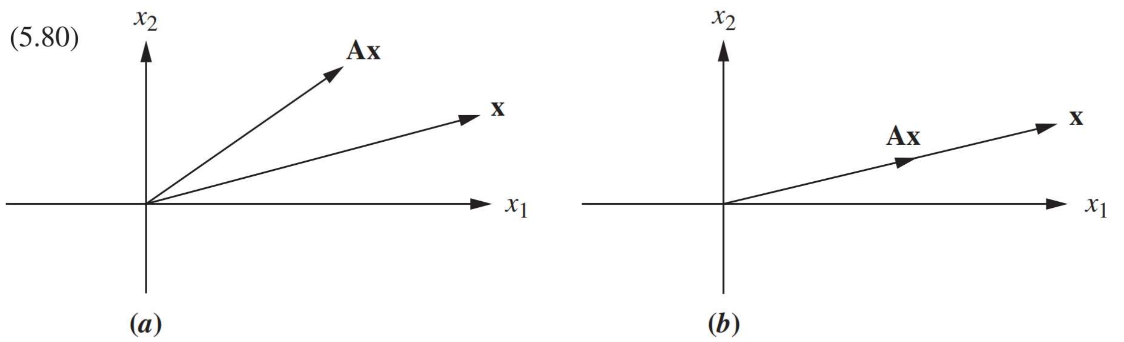

FIGURE 5.32 To be an eigenvector, the transformation $\mathbf{Ax}$ must be collinear with $\mathbf{x}$; thus, in (*a*), $\mathbf{x}$ is not an eigenvector; in (*b*), it is.

Eigenvalue. The eigenvalues of the matrix $\mathbf{A}$ are the values of λ_i that satisfy Eq. (5.80) for $\mathbf{x_i} \neq \mathbf{0}$.

To find the eigenvectors, we rearrange Eq. (5.80). Eigenvectors, $\mathbf{x_i}$, satisfy

$$\mathbf{0} = (\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x_i} \tag{5.81}$$

Solving for $\mathbf{x_i}$ by premultiplying both sides by $(\lambda_i \mathbf{I} - \mathbf{A})^{-1}$ yields

$$\mathbf{x_i} = (\lambda_i \mathbf{I} - \mathbf{A})^{-1}\mathbf{0} = \frac{\operatorname{adj}(\lambda_i \mathbf{I} - \mathbf{A})}{\det(\lambda_i \mathbf{I} - \mathbf{A})}\mathbf{0} \tag{5.82}$$

Since $\mathbf{x_i} \neq \mathbf{0}$, a nonzero solution exists if

$$\boxed{\det(\lambda_i \mathbf{I} - \mathbf{A}) = \mathbf{0}} \tag{5.83}$$

from which λ_i, the eigenvalues, can be found.

- Diagonal representation
  - Equivalent to partial fraction expansion of t.f.
  - Use eigenvectors to form a transform matrix **P**

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x_n}]$$

$$\mathbf{A}\mathbf{x_i} = \lambda_i \mathbf{x_i}$$

$$\mathbf{AP} = \mathbf{PD}$$

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$$

$$\boxed{\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu}}$$

$$\boxed{\mathbf{y} = \mathbf{CPz} + \mathbf{Du}}$$

Left **D** is diagonal matrix with eigenvalues, while right **D** is the through-input matrix, often $\mathbf{D} = 0$

Diagonal state space representation

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Diagonalizing a System in State Space

PROBLEM: Given the system of Eqs. (5.94), find the diagonal system that is similar.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u \tag{5.94a}$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x} \tag{5.94b}$$

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\mathbf{C}\mathbf{P} = \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 5 & -1 \end{bmatrix}\mathbf{z}$$

$$\boxed{\mathbf{z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}}$$

$$\boxed{\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u}}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| = \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} = \lambda^2 + 6\lambda + 8$$

$$\mathbf{A}\mathbf{x_i} = \lambda\mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Diagonal state space representation

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PROBLEM: For the system represented in state space as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}u$$
$$y = \begin{bmatrix} 1 & 4 \end{bmatrix}\mathbf{x}$$

convert the system to one where the new state vector, $\mathbf{z}$, is

$$\mathbf{z} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\mathbf{x}$$

ANSWER:

$$\dot{\mathbf{z}} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -3 \\ -11 \end{bmatrix}u$$
$$y = \begin{bmatrix} 0.8 & -1.4 \end{bmatrix}\mathbf{z}$$

PROBLEM: For the original system find the diagonal system that is similar.

ANSWER:

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}u$$
$$y = \begin{bmatrix} -2.121 & 2.6 \end{bmatrix}\mathbf{z}$$

Solution of the State Equations

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$
$$= \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathscr{L}^{-1}\left[\frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right] = \mathbf{\Phi}(t)$$

Solution of the State Equations

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

for scalar case

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s).$$

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)d\tau.$$

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) = \frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathscr{L}^{-1}[(s\mathbf{I}-\mathbf{A})^{-1}] = \mathscr{L}^{-1}\left[\frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\right] = \mathbf{\Phi}(t)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

zero-input response

Zero-state response as convolution integral

*The **matrix exponential** function is defined by in a similar Taylor series form*

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \cdots,$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

matrix exponentials is a DEFINITION

$$X(t) = e^{tA} = \sum_{i=0}^{\infty} \frac{1}{i!} [tA]^i .$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Proof:

- *Uniqueness of Solutions*

If we have two solutions $x_1, x_2 : \mathbb{R} \to \mathbb{R}^{n_x}$, then $y = x_1 - x_2$ satisfies

$$\dot{y}(t) = Ay(t) \quad \text{with} \quad y(0) = 0 .$$

The auxiliary function $v(t) = e^{-At}y(t)$ satisfies

$$\begin{aligned} \dot{v}(t) &= -Ae^{-At}y(t) + e^{-At}Ay(t) = -Ae^{-At}y(t) + Ae^{-At}y(t) = 0 \\ v(0) &= 0 , \end{aligned}$$

$$\Longrightarrow \quad v(t) = y(t) = 0 \quad \Longrightarrow \quad x_1 = x_2 .$$

Solution of the State Equations

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

matrix exponentials is a DEFINITION

$$X(t) = e^{tA} = \sum_{i=0}^{\infty} \frac{1}{i!}[tA]^i .$$

Proof:

- *Verify the ODE*

Generalized Leibniz integral rule.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} g(t,\tau)\,\mathrm{d}\tau = g(t,b(t))\,\dot{b}(t) - g(t,a(t))\,\dot{a}(t) + \int_{a(t)}^{b(t)} g_t(t,\tau)\,\mathrm{d}\tau .$$

$$\begin{aligned}\dot{x}(t) &= A\,e^{At}x(0) + e^{A(t-t)}Bu(t) + \int_0^t A\,e^{A(t-\tau)}Bu(\tau)d\tau \\ &= A\left[e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] + Bu(t) \\ &= Ax(t) + Bu(t)\end{aligned}$$

Solution of the State Equations

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Specially, the solution of an unforced system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$$

The matrix exponential function describes the unforced response of the system and is called the fundamental or state transition matrix $\Phi(t, 0)$.

Thus, the general solution can be written as

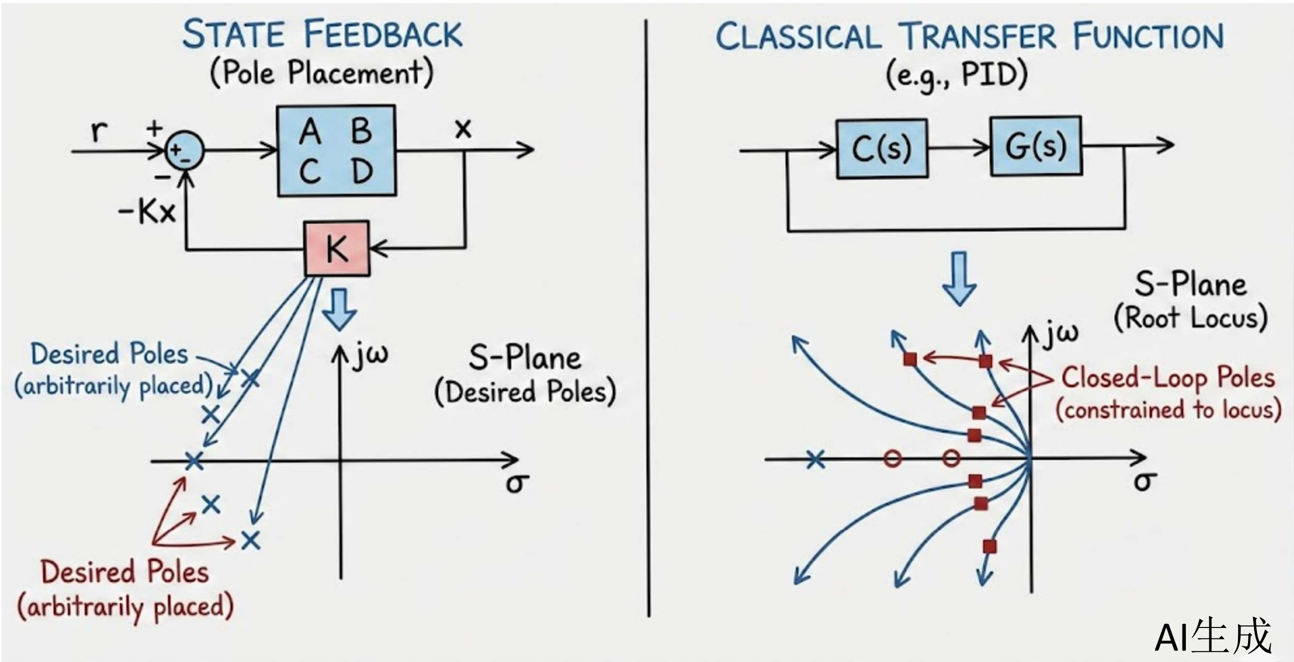
$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau.$$

NOTE, up to now, we are talking about LTI system, for nonlinear or time-varying, there is NO nice general solution form.

State Feedback for Pole Placement

Motivation for Pole Placement

- Frequency domain methods of design do not allow us to specify all poles in systems of order higher than 2 because they do not allow for a sufficient number of unknown parameters to place all of the closed-loop poles uniquely.
  - One gain to adjust, or compensator pole and zero to select, does not yield a sufficient number of parameters to place all the closed-loop poles at desired locations.
  - Remember, to place *n* unknown quantities, you need *n* adjustable parameters.



Comparison of closed-loop pole placement flexibility between State Feedback and Classical approaches.

Concept of State Feedback

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- Ideally, if the n states of the system are available for feedback, we can introduce n adjustable parameters in the gain matrix K.

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x} + r) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}r} \quad (12.3a)$$

$$\boxed{y = \mathbf{C}\mathbf{x}} \quad (12.3b)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$\mathbf{C} = [\, c_1 \quad c_2 \quad \cdots \quad c_n \,]$$

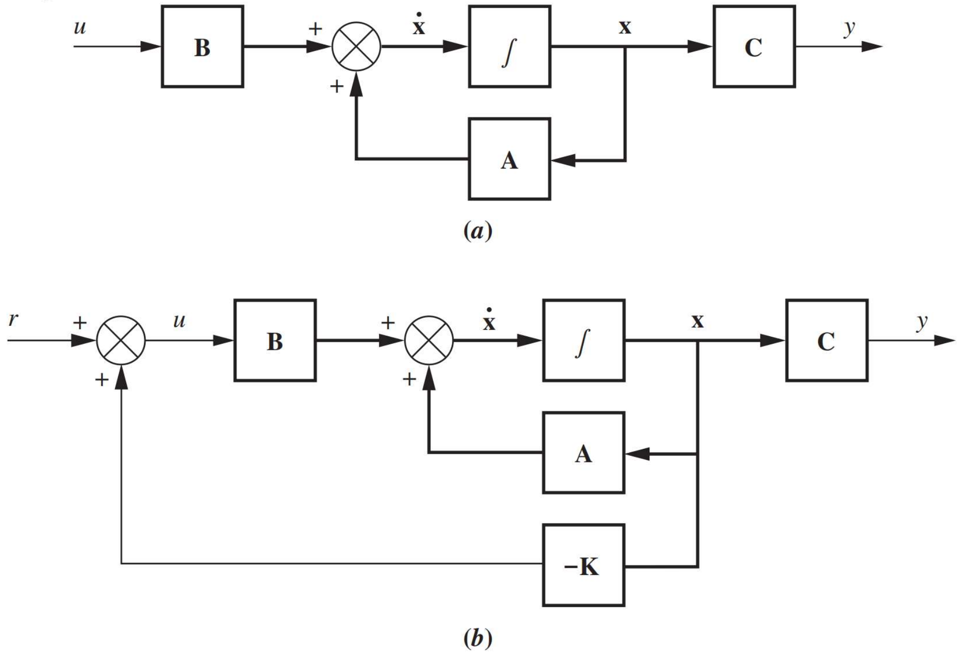


FIGURE 12.2 **a.** State-space representation of a plant; **b.** plant with state-variable feedback

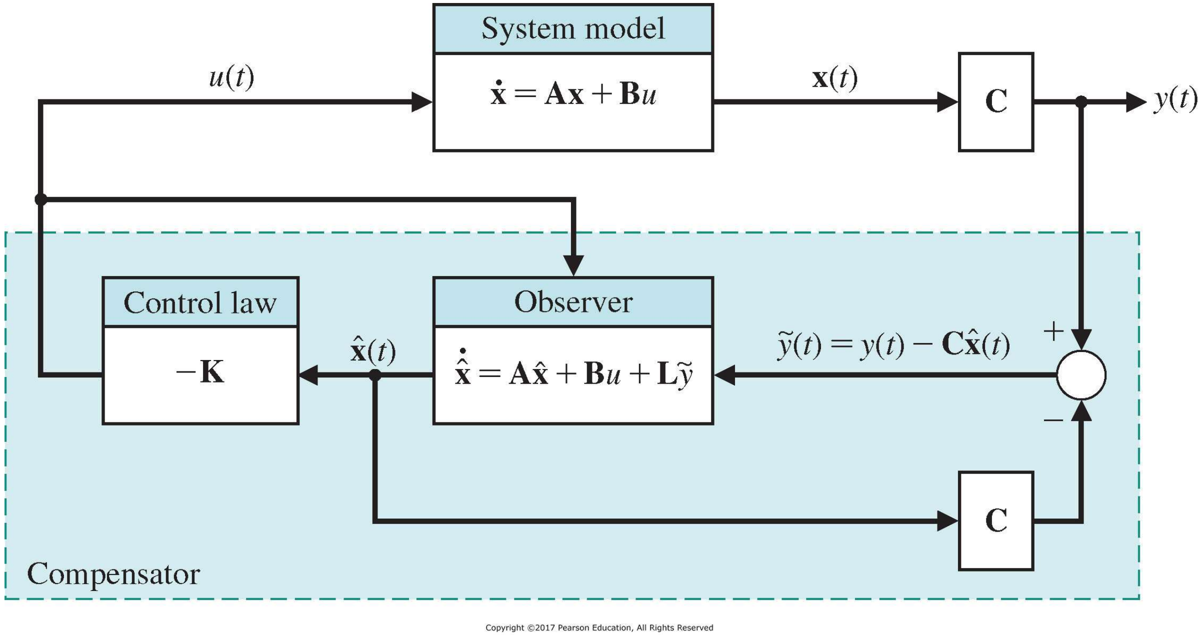
Concept of Output Feedback Design

- ***State feedback design using only output meausrement typically comprises three steps:***

1 we assume that all the state variables are measurable and utilize them in a full-state feedback control law.

2 to construct an observer to estimate the states that are not directly measured and available as outputs.

*3 is to appropriately connect the observer to the full-state feedback control law. For linear system, there is a so-called **separation principle** such that the output feedback system just works*



Controllability and Observability

Controllability

A key question that arises in the design of state variable compensators is whether or not all the poles of the closed-loop system can be arbitrarily placed in the complex plane.

if the system is controllable and observable, then we can.

The concepts of controllability and observability were introduced by Kalman in the 1960s:

A system is completely controllable if there exists an unconstrained control $\mathbf{u}(t)$ that can transfer any initial state $\mathbf{x}(t_0)$ to any other desired location $\mathbf{x}(t)$ in a finite time, $t_0 \leq t \leq T$.

For the SISO LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

we can determine whether the system is controllable by examining the algebraic condition

$$\text{rank}[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{n-1}\mathbf{B}] = n.$$

dimension of the system.

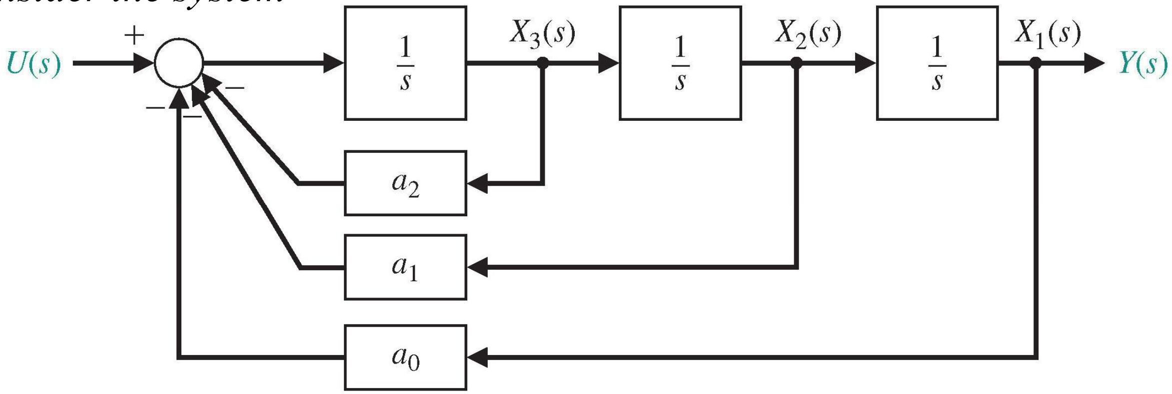
The controllability matrix P_c is described in terms of A and B as

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{n-1}\mathbf{B}],$$

If the determinant of P_c is nonzero, the system is controllable.

Controllability

Example: *Let us consider the system*



Check whether the system is controllable or not?

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t).$$

$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t) + [0]u(t)$$

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & a_2^2 - a_1 \end{bmatrix}.$$

The determinant of Pc = -*1, hence this system is controllable.*

Observability

Observability refers to the ability to estimate a state variable.

A system is completely observable if and only if there exists a finite time T such that the initial state $\mathbf{x}(0)$ can be determined from the observation history $y(t)$ given the control $\mathbf{u}(t)$, $0 \leq t \leq T$.

For the SISO LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

This system is completely observable when the determinant of the observability matrix Po is nonzero,

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix},$$

Supplementary:

1. Advanced state variable design techniques can handle situations wherein the system is not completely controllable, but where the states (or linear combinations thereof) that cannot be controlled are inherently stable. These systems are classified as stabilizable.

2. These same techniques can handle cases wherein the system is not completely observable, but where the states (or linear combinations thereof) that cannot be observed are inherently stable. These systems are classified as detectable.

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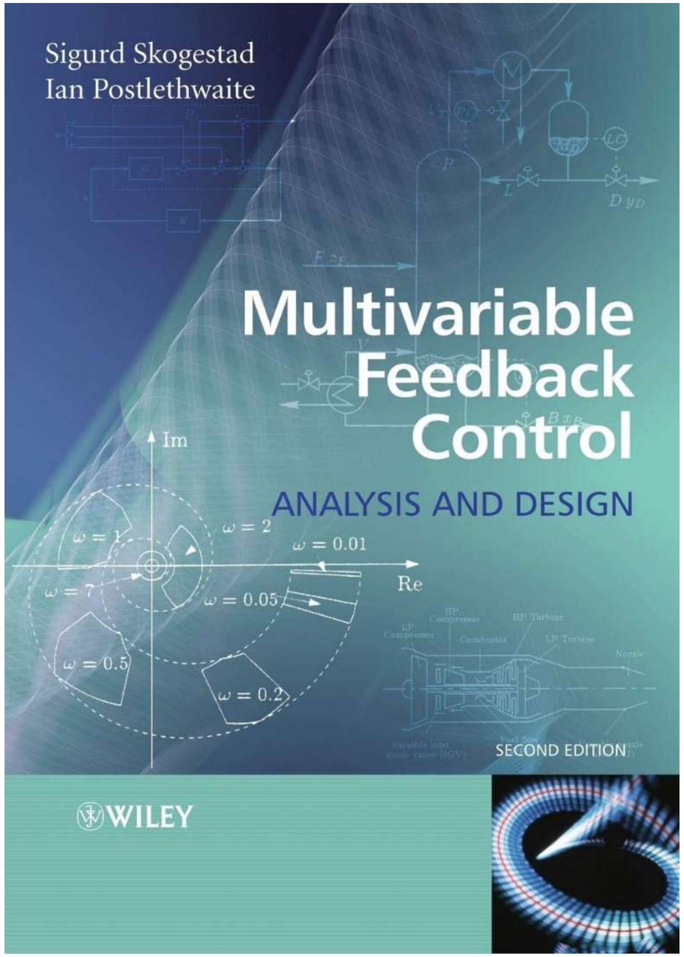
The Bible for
Modern (State Space) Control System

Bible Recommendation



4.2 State controllability and state observability

Definition 4.1 State controllability *The dynamical system* $\dot{x} = Ax + Bu$ *or the pair* (A, B) *is said to be state controllable if, for any initial state* $x(0) = x_0$*, any time* $t_1 > 0$ *and any final state* x_1*, there exists an input* $u(t)$ *such that* $x(t_1) = x_1$*. Otherwise the system is said to be state uncontrollable.*

From (4.7) one can verify that a particular input which achieves $x(t_1) = x_1$ is

$$u(t) = -B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}(e^{At_1}x_0 - x_1) \tag{4.40}$$

where $W_c(t)$ is the Gramian matrix at time t,

$$W_c(t) \triangleq \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau \tag{4.41}$$

There are many ways to check whether a system is state controllable. First, we have that (A, B) is state controllable if and only if the controllability matrix

$$\mathcal{C} \triangleq [\, B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B \,] \tag{4.42}$$

has rank n (full row rank). Here n is the number of states.

Controller Design via State Feedback

Full-state Feedback Control Design

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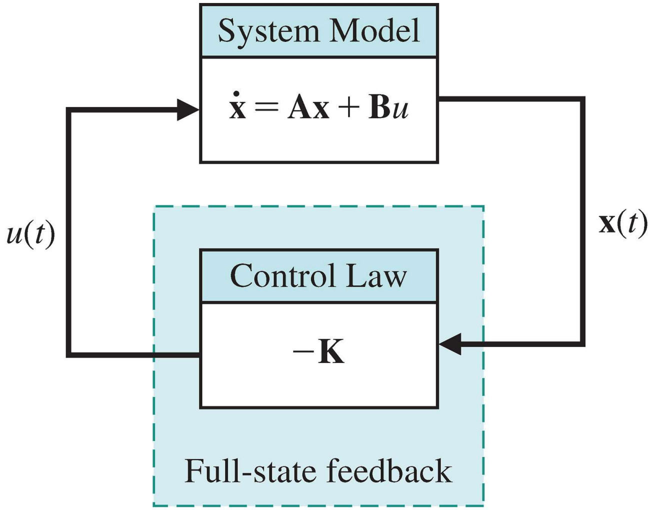
Assume that all the states are available for feedback, we have

dimension of K?

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Determining the gain matrix K is the objective of the full-state feedback design procedure.

achieve the desired pole locations of the closed-loop system.



Theorem : Given the pair (A, B), we can always determine K to place all the system closed-loop poles in the left half-plane if and only if the system is completely controllable—that is, if and only if the controllability matrix Pc is full rank.

Full-state Feedback Control Design

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Assume that all the states are available for feedback, we have

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Determining the gain matrix K is the objective of the full-state feedback design procedure.

achieve the desired pole locations of the closed-loop system.

we find the closed-loop system to be

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t).$$

The characteristic equation

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})) = 0.$$

If all the roots of the characteristic equation lie in the left half-plane, then the closed-loop system is stable.

*This is known as the **regulator problem**. That is, we want to compute K so that all initial conditions are driven to zero in a specified fashion.*

In other words, for any initial condition, it follows that

$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{B}\mathbf{K})t}\mathbf{x}(t_0) \rightarrow 0 \qquad \text{as } t \rightarrow \infty.$$

Full-state Feedback Control Design

Assume that all the states are available for feedback, we have

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Determining the gain matrix K is the objective of the full-state feedback design procedure.

For tracking purpose, addition of a reference input can be written as

$$u(t) = -\mathbf{K}\mathbf{x}(t) + Nr(t),$$

where r(t) is the reference input.

Example: *Let us consider the third-order system*

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = u(t).$$

We can select the state variables as

$$x_1(t) = y(t),$$

$$x_2(t) = dy(t)/dt,$$

$$x_3(t) = d^2y(t)/dt^2,$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t).$$

Controller Design by Matching Coefficients

Example: *We design a state feedback controller as*

$$u(t) = -\mathbf{K}\mathbf{x}(t), \qquad \mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

then the closed-loop system is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t).$$

The state feedback matrix is

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2-k_1 & -3-k_2 & -5-k_3 \end{bmatrix},$$

and the characteristic equation is

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})) = \lambda^3 + (5 + k_3)\lambda^2 + (3 + k_2)\lambda + (2 + k_1) = 0.$$

If we seek a rapid response with a low overshoot, we choose a desired characteristic equation such as

$$\Delta(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda + \zeta\omega_n).$$

Controller Design by Matching Coefficients

Example:

$$\Delta(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda + \zeta\omega_n).$$

We choose $\xi = 0.8$ *for minimal overshoot and* ω_n *to meet the settling time requirement*

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.8)\omega_n} \approx 1. \longrightarrow \omega_n = 6$$

the desired characteristic equation is

$$(\lambda^2 + 9.6\lambda + 36)(\lambda + 4.8) = \lambda^3 + 14.4\lambda^2 + 82.1\lambda + 172.8.$$

Recall the characteristic equation to be designed

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = \lambda^3 + (5 + k_3)\lambda^2 + (3 + k_2)\lambda + (2 + k_1) = 0.$$

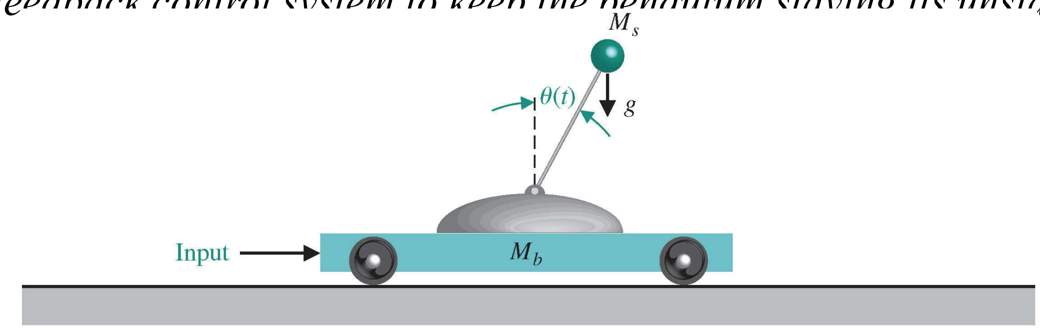
yields the three equations

$$\begin{aligned} 5 + k_3 &= 14.4 \\ 3 + k_2 &= 82.1 \\ 2 + k_1 &= 172.8. \end{aligned} \longrightarrow K = [170.8, 79.1, 9.4]$$

Controller Design by Matching Coefficients

Example: *Inverted pendulum control*

Consider the control of an unstable inverted pendulum balanced on a moving cart. Now we tend to design a suitable state variable feedback control system to keep the pendulum staying its unstable position.



To simplify, we assume that the control input, u(t), is an acceleration signal, ***we can focus on the unstable dynamics of the pendulum.***

$$\ddot{\theta}(t) = \frac{g}{l}\theta(t) - \frac{1}{l}u(t).$$

Let the state vector be $(x_1(t), x_2(t)) = (\theta(t), \dot{\theta}(t)).$

The A matrix has the characteristic equation $\lambda^2 - \frac{g}{l} = 0$ *with one root in the right-hand s-plane.*

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{bmatrix} 0 \\ -1/l \end{bmatrix}u(t).$$

Controller Design by Matching Coefficients

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Example: *Inverted pendulum control*

To stabilize the system, we generate a control signal

$$u(t) = -\mathbf{Kx}(t) = -[k_1 \quad k_2]\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -k_1x_1(t) - k_2x_2(t).$$

Substituting this control signal relationship into the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ (g + k_1)/l & k_2/l \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Obtaining the characteristic equation, we have

$$\begin{bmatrix} \lambda & -1 \\ -(g + k_1)/l & \lambda - k_2/l \end{bmatrix} = \lambda\left(\lambda - \frac{k_2}{l}\right) - \frac{g + k_1}{l} = \lambda^2 - \left(\frac{k_2}{l}\right)\lambda + \frac{g + k_1}{l}.$$

Thus, for the system to be stable, we require that

$$k_2/l < 0 \text{ and } k_1 > -g.$$

If we wish to achieve a rapid response with modest overshoot, we select

$$\omega_n = 10 \text{ and } \zeta = 0.8.$$

Then we require

$$\frac{k_2}{l} = -16 \quad \text{and} \quad \frac{k_1 + g}{l} = 100.$$

Controller Design by Matching Coefficients

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Tips for making our life easier:

1. *For a SISO LTI, Given the desired characteristic equation*

$$q(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_o,$$

Then Ackermann's formula is useful for determining the state variable feedback matrix

$$\mathbf{K} = [0 \quad 0 \ldots 0 \quad 1]\mathbf{P}_c^{-1} q(\mathbf{A}),$$

Invertible Matrix

where Pc is the controllability matrix and

$$q(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots \alpha_1\mathbf{A} + \alpha_0\mathbf{I},$$

2. Matlab Code:

Feedback gain matrix **K**.

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

Vector containing desired closed-loop poles.

K=acker(A,B,P)
K=place(A,B,p)

Controllability matrix $\quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, $y = \mathbf{C}\mathbf{x} + \mathbf{D}u$

Pc=ctrb(A,B)

Po=obsv(A,C)

Observability matrix $\quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, $y = \mathbf{C}\mathbf{x} + \mathbf{D}u$

Controller Design by Transformation

- We find the phase variable version of a controllable system, and design the gain matrix.
  - It can be proved that the transformation matrix **P** is related to the two controllability matrices:

$$\boxed{\mathbf{P} = \mathbf{C_{Mz}}\mathbf{C_{Mx}}^{-1}}$$

- The gain matrix for the original system can be derived using the transformation matrix **P**.

$$\mathbf{K_z} = \mathbf{K_x}\mathbf{P}^{-1}$$

$$\dot{\mathbf{z}} = \mathbf{A_z}\mathbf{z} + \mathbf{B_z}u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C_z}\mathbf{z} = [-1 \quad 1 \quad 0]\,\mathbf{z}$$

Open loop poles

$$\dot{\mathbf{x}} = \mathbf{A_x}\mathbf{x} + \mathbf{B_x}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = [4 \quad 1 \quad 0]\,\mathbf{x}$$

Characteristic equation coefficients

PROBLEM: Design a state-variable feedback controller to yield a 20.8% overshoot and a settling time of 4 seconds for a plant,

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} \tag{12.43}$$

that is represented in cascade form as shown in Figure 12.9.

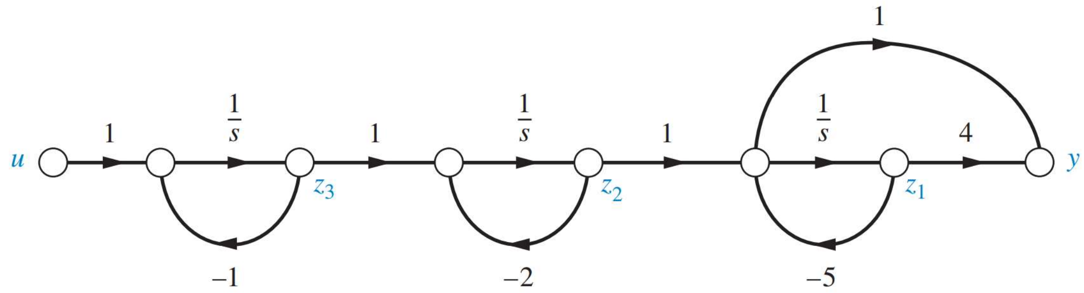

FIGURE 12.9 Signal-flow graph for plant of Example 12.4

Controller Design by Transformation

PROBLEM: Design a linear state-feedback controller to yield 20% overshoot and a settling time of 2 seconds for a plant

$$G(s) = \frac{(s+6)}{(s+9)(s+8)(s+7)}$$

that is represented in state space in cascade form by

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & -9 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{z} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\mathbf{z}$$

ANSWER:

Partially Controllable System

Partially Controllable System

- Given a single-input continuous-time linear time-invariant controlled system, determine whether there is a state feedback matrix K so that the closed-loop eigenvalues are configured to the following positions.

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$(a)\lambda_1^* = -2, \lambda_2^* = -2, \lambda_3^* = -2, \lambda_4^* = -2$$

$$(b)\lambda_1^* = -3, \lambda_2^* = -3, \lambda_3^* = -3, \lambda_4^* = -2$$

$$(c)\lambda_1^* = -3, \lambda_2^* = -4, \lambda_3^* = -3, \lambda_4^* = -3$$

Partially Controllable System

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- Write out the differential equations for x_3 and x_4:
  - $\dot{x}_3 = -2x_3 + 1u$
  - $\dot{x}_4 = -2x_4 + 1u$
  - Define a new state denoted by $\Delta x = x_3 - x_4$,
  - which is governed by $\frac{\mathrm{d}}{\mathrm{dt}}\Delta x = -2\Delta x$
  - The eigenvalue of the new state Δx cannot be modified by u.
    - This is what essentially the concept of uncontrollable state.
- For the first block, we can use the idea of back-stepping:
  - To control x_1 or do pole placement, we can use x_2 as virtual input v.
  - That is, $x_2 = \frac{1}{s}(-2x_2 + u) = v$
  - If we want to place the first eigenvalue to -12, then we can design $v = -14x_1$,
  - which gives $\dot{x}_1 = 2x_1 + v = -12x_1$
  - Even though the DoF of $x_2 = -14x_1$ has been used, its eigenvalue is not yet been determined.
  - We can design $u = -14x_2$ to place the eigenvalue of x_2 to a new location.
  - $\dot{x}_2 = 2x_2 + 1u = -12x_2$.
  - The derivative of x_2 is related to its eigenvalue. The value of $x_2 = v$ is used to place the eigenvalue of x_1.

- Write out the differential equations for x_3 and x_4:
  - $\dot{x}_3 = -2x_3 + 1u$
  - $\dot{x}_4 = -2x_4 + 1u$
  - Define a new state denoted by $\Delta x = x_3 - x_4$,
  - which is governed by $\frac{\mathrm{d}}{\mathrm{dt}}\Delta x = -2\Delta x$
  - The eigenvalue of the new state Δx cannot be modified by u.
    - This is what essentially the concept of uncontrollable state.
- For the first block, we can use the idea of back-stepping:
  - To control x_1 or do pole placement, we can use x_2 as virtual input v.
  - That is, $x_2 = \frac{1}{s}(-2x_2 + u) = v$
  - If we want to place the first eigenvalue to -12, then we can design $v = -14x_1$,
  - which gives $\dot{x}_1 = 2x_1 + v = -12x_1$
  - Even though the DoF of $x_2 = -14x_1$ has been used, its eigenvalue is not yet been determined.
  - We can design $u = -14x_2$ to place the eigenvalue of x_2 to a new location.
  - $\dot{x}_2 = 2x_2 + 1u = -12x_2$.
  - The derivative of x_2 is related to its eigenvalue. The value of $x_2 = v$ is used to place the eigenvalue of x_1.

Partially Controllable System

$$\dot{x} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} u$$

- Further consider this state space model and do what we have done again.
- Write out the differential equation for x_3 and x_4:
  - $\dot{x}_3 = -2x_3 + 1u$
  - $\dot{x}_4 = -2x_4 + 2u$
  - Define a new state denoted by $\Delta x = 2x_3 - x_4$,
  - which is governed by $\frac{\mathrm{d}}{\mathrm{dt}}\Delta x = -4x_3 + 2x_4 = -2\Delta x$
  - The eigenvalue of the new state Δx cannot be modified by u.
    - This is again what essentially the concept of uncontrollable state.
- For the first block, let's also cancel the input u by defining a new delta state.
  - $\frac{\mathrm{d}}{\mathrm{dt}}\Delta x = \frac{\mathrm{d}}{\mathrm{dt}}(x_1 - x_2) = 3x_1 + x_2 - 3x_2 = -3\Delta x + x_2$
  - Here, x_2 is affected by u and can be treated as a virtual input.

Partially Controllable System

- Finally, let's also check the eigenvalues of 3 and -2 they are not closed dynamics (--closed dynamics cannot be affected by input u)
  - $\dot{x}_2 = 3x_2 + 1u$
  - $\dot{x}_3 = -2x_3 + 1u$
  - $\frac{\mathrm{d}}{\mathrm{dt}}\Delta x = 3x_2 + 2x_3 = 3\Delta x + 5x_3$
  - Now the state x_3 can be viewed as a virtual input that is affected by input u.

$$\dot{x} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} u$$

- The reason why a matrix's determinant is equal to the product of all its eigenvalues.

对于约当标准型矩阵（Jordan Canonical Form, JCF），其行列式的计算非常简单：

约当标准型矩阵的行列式等于其主对角线上所有元素的乘积。

如果一个约当标准型矩阵 J 的主对角线元素为 $\lambda_1, \lambda_2, \ldots, \lambda_n$（这些就是矩阵的特征值），那么：

$$\det(J) = \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n$$

为什么可以这样计算？

这个结论基于两个基本的线性代数性质：

1. **三角矩阵的行列式性质：** 约当标准型矩阵是一个**上三角矩阵**（Upper Triangular Matrix）。它的非零元素只出现在主对角线以及主对角线上方的一条次对角线上，主对角线下方全为零。
   - 对于任何上三角矩阵（或下三角矩阵），其行列式都等于主对角线上所有元素的乘积。
   - **直观理解：** 在计算行列式时，可以使用拉普拉斯展开（Laplace Expansion）。对于上三角矩阵，如果我们沿着第一列展开，只需要考虑 a_{11} 这一项，因为它乘以的代数余子式仍然是一个上三角矩阵。以此类推，最终结果就是对角线元素的乘积。
2. **相似矩阵的行列式性质：** 任何方阵 A 都相似于其约当标准型 J。这意味着存在一个可逆矩阵 P，使得 $A = PJP^{-1}$。
   - 根据行列式的乘法性质：$\det(AB) = \det(A)\det(B)$。
   - 因此，$\det(A) = \det(PJP^{-1}) = \det(P)\det(J)\det(P^{-1})$。
   - 由于 $\det(P^{-1}) = 1/\det(P)$，它们可以相互抵消。
   - 结论：$\det(A) = \det(J)$。

总结： 因为约当标准型 J 是一个上三角矩阵，所以它的行列式等于对角线元素（特征值）的乘积。又因为原矩阵 A 与 J 相似，所以它们的行列式相等。这就是为什么矩阵 A 的行列式也等于其所有特征值乘积的原因。

Observer

Observer Design

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Next big question,

- *what if the state x(t) is not available?*
- *Especially when x(t) does not have a good physical meaning.*
- *Or when you can afford an expensive sensor?*

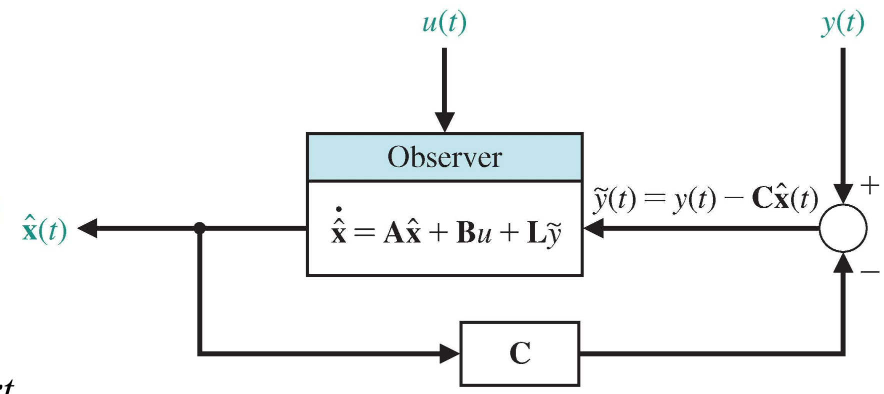
*Fortunately, **if the system is completely observable with a given set of outputs**, then it is possible to determine (or to estimate) the states that are not directly measured (or observed).*

According to Luenberger , the full-state observer for the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))$$



where

- $\hat{x}(t)$ *is the so-called estimates of the state* $x(t)$
- *matrix L is the observer gain matrix and is to be determined as part of the observer design procedure.*

Observer Design

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)) \qquad \hat{x}(t_0) = \hat{x}_0$$

in general, not equal x_0

The goal of the observer is to provide an estimate $\hat{x}(t) \to x(t)$ as $t \to \infty$.

Define the observer estimation error as

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t).$$

The observer design should produce an observer with the property that

$$e(t) \to 0 \text{ as } t \to \infty$$

Taking the time-derivative of the estimation error

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).\end{aligned}$$

Conclusion: We can achieve our goal if the characteristic equation

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})) = 0$$

has all its roots in the left half-plane.

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Example: *Consider the second-order system*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t).$$

In this example, we can only directly observe the state $y(t) = x_1(t)$.

checking the system observability

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

The full-state observer for the system

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))$$

where

$$\mathbf{L} = [L_1 \quad L_2]^T.$$

Then the characteristic equation of the estimation error yields

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = \lambda^2 + (L_1 - 6)\lambda - 4(L_1 - 2) + 3(L_2 + 1),$$

Observer Design

what does this "settling" mean?

Suppose that the desired characteristic equation is given by

$$\Delta_d(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2.$$

We can select $\xi = 0.8$ *and* $\omega_n = 10$*, resulting in an expected settling time of less than 0.5 second.*

Equating the coefficients

$$L_1 - 6 = 16$$

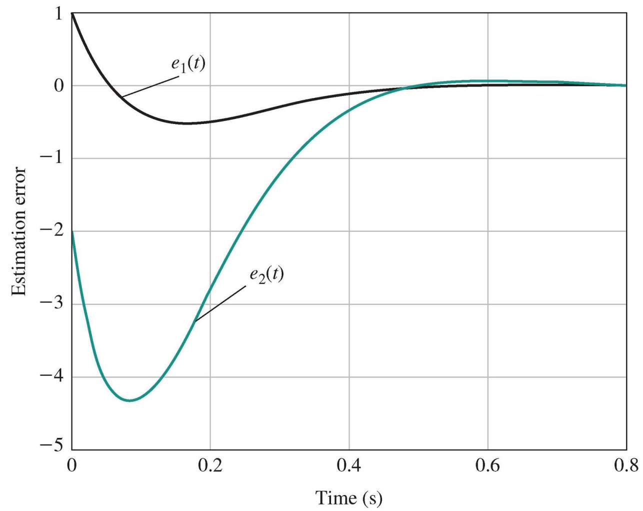
$$-4(L_1 - 2) + 3(L_2 + 1) = 100$$

which, when solved, produces

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} 22 \\ 59 \end{bmatrix}.$$

The observer is thus given by

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\hat{\mathbf{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) + \begin{bmatrix} 22 \\ 59 \end{bmatrix}(y(t) - \hat{x}_1(t)).$$



Observer Design

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Note:

1. *Given the desired observer characteristic equation*

$$p(\lambda) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0.$$

Ackermann's formula can also be employed to place the roots of the observer

$$\mathbf{L} = p(\mathbf{A})\mathbf{P}_o^{-1}[0 \cdots 0 \quad 1]^T,$$

where Po is the observability matrix

$$p(\mathbf{A}) = \mathbf{A}^n + \beta_{n-1}\mathbf{A}^{n-1} + \cdots + \beta_1\mathbf{A} + \beta_0\mathbf{I}.$$

2. *Up to now, the effectiveness of the observer has NOTHING to do with the control input and it will NOT alter the behaviour of the system*

Observer based Feedback Control

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Recall

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

It seems reasonable that we can employ the state estimate in the feedback control law in place of $x(t)$.

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t).$$

We need to verify that, using the estimate still retain the stability of the closed-loop system.

Proof: *Substituting the observer-based feedback law into the system model yields*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t),$$

and with $\hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{e}(t)$ *we obtain*

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t).$$

Recall the the estimation error

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).$$

$$\Rightarrow \begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{pmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{pmatrix}.$$

$$\Rightarrow \Delta(\lambda) = \det\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})\right)\det\left(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})\right)$$

So if A-BK and A-LC are both Hurwitz, then the overall system is stable.

Observer based Feedback Control

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The fact that the full-state feedback law and the observer can be designed independently is an illustration of ***the separation principle.***

The design procedure is summarized as follows:

1. Determine $\mathbf{K}$ such that $\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = 0$ has roots in the left half-plane and place the poles appropriately to meet the control system design specifications. The ability to place the poles arbitrarily in the complex plane is guaranteed if the system is completely controllable.

2. Determine $\mathbf{L}$ such that $\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = 0$ has roots in the left half-plane and place the poles to achieve acceptable observer performance. The ability to place the observer poles arbitrarily in the complex plane is guaranteed if the system is completely observable.

3. Connect the observer to the full-state feedback law using

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t).$$

Compensator Transfer Function.

$$U(s) = [-\mathbf{K}(s\mathbf{I} - (\mathbf{A} - \mathbf{BK} - \mathbf{LC}))^{-1}\mathbf{L}]Y(s).$$

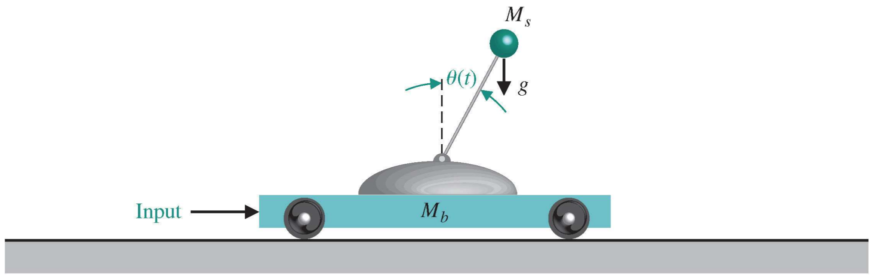
This controller is also commonly referred to as a ***stabilizing controller.***

Note that, even though A - BK is stable and A - LC is stable, it does not necessarily follow that A - BK - LC is stable.

Observer based Feedback Control

Example: *Compensator design for the inverted pendulum*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix} u(t),$$



$$y(t) = [1 \quad 0 \quad 0 \quad 0]\mathbf{x}(t).$$

Let the system parameters be

$$l = 0.098 \text{ m}, \ g = 9.8 \text{ m/s}^2, \ m = 0.825 \text{ kg} \quad M = 8.085 \text{ kg}.$$

Checking controllability yields the controllability matrix

$$\mathbf{P}_c = \begin{bmatrix} 0 & 0.1237 & 0 & 1.2621 \\ 0.1237 & 0 & 1.2621 & 0 \\ 0 & -1.2621 & 0 & -126.21 \\ -1.2621 & 0 & -126.21 & 0 \end{bmatrix}.$$

$$\mathbf{P}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Observer based Feedback Control

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Step 1: Design the Full-State Feedback Control Law

The open-loop system poles are located at

$$\lambda = 0, 0, -10, \text{ and } 10,$$

hence the open-loop system is unstable (there is a pole in the right half-plane).

$$q(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda^2 + a\lambda + b),$$

To obtain a settling time less than 10 seconds with low overshoot, we can select

$$(\zeta, \omega_n) = (0.8, 0.5).$$

Then, we choose a separation factor of 20 between the dominant poles and the remaining poles, from which it follows that

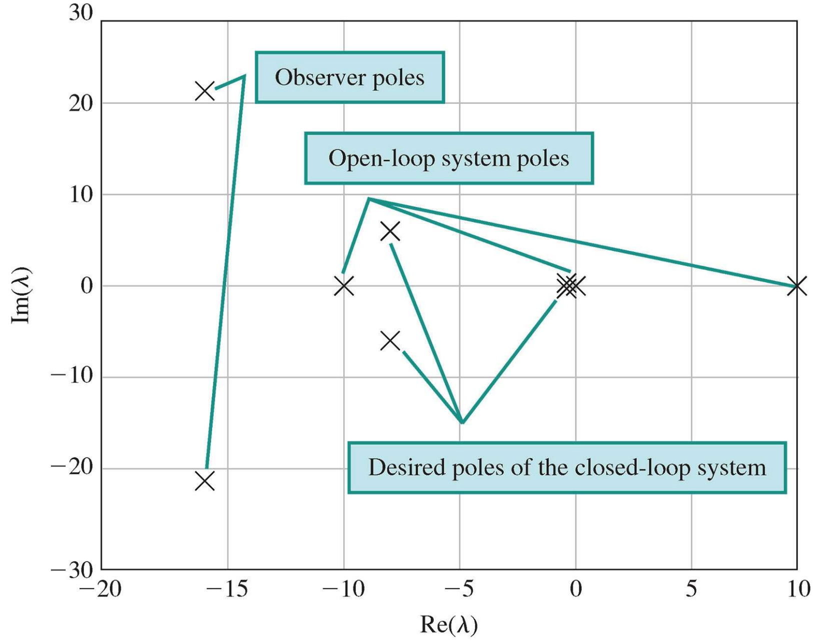
$$(a, b) = (16, 100)$$

The desired roots are

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = (\lambda + 8 \pm j6)(\lambda + 0.4 \pm j0.3).$$

Using Ackermann's formula yields the feedback gain matrix

$$\mathbf{K} = [-2.2509 \quad -7.5631 \quad -169.0265 \quad -14.0523].$$



Observer based Feedback Control

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Step 2: Observer Design

The desired observer characteristic equation is selected to be of the form

$$p(\lambda) = (\lambda^2 + c_1\lambda + c_2)^2$$

where

$$c_1 = 32 \text{ and } c_2 = 711.11.$$

These values should produce a response to an initial state estimation error that settles in less than 0.5 second with minimal percent overshoot.

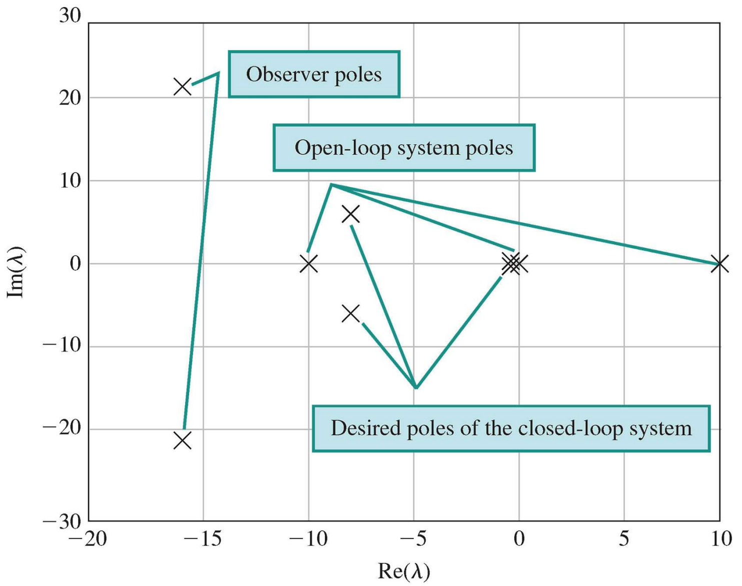
Using Ackermann's formula, we have

$$\mathbf{L} = \begin{bmatrix} 64.0 \\ 2546.22 \\ -5.1911\text{E}04 \\ -7.6030\text{E}05 \end{bmatrix}.$$

achieves the desired observer pole locations

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = ((\lambda + 16 + j21.3)(\lambda + 16 - j21.3))^2$$

The goal is to achieve an accurate estimate as fast as possible without resulting in too large a gain matrix L. How large is too large depends on the problem under consideration. The trade-off between the time required to obtain accurate observer performance and the amount of noise amplification is a primary design issue



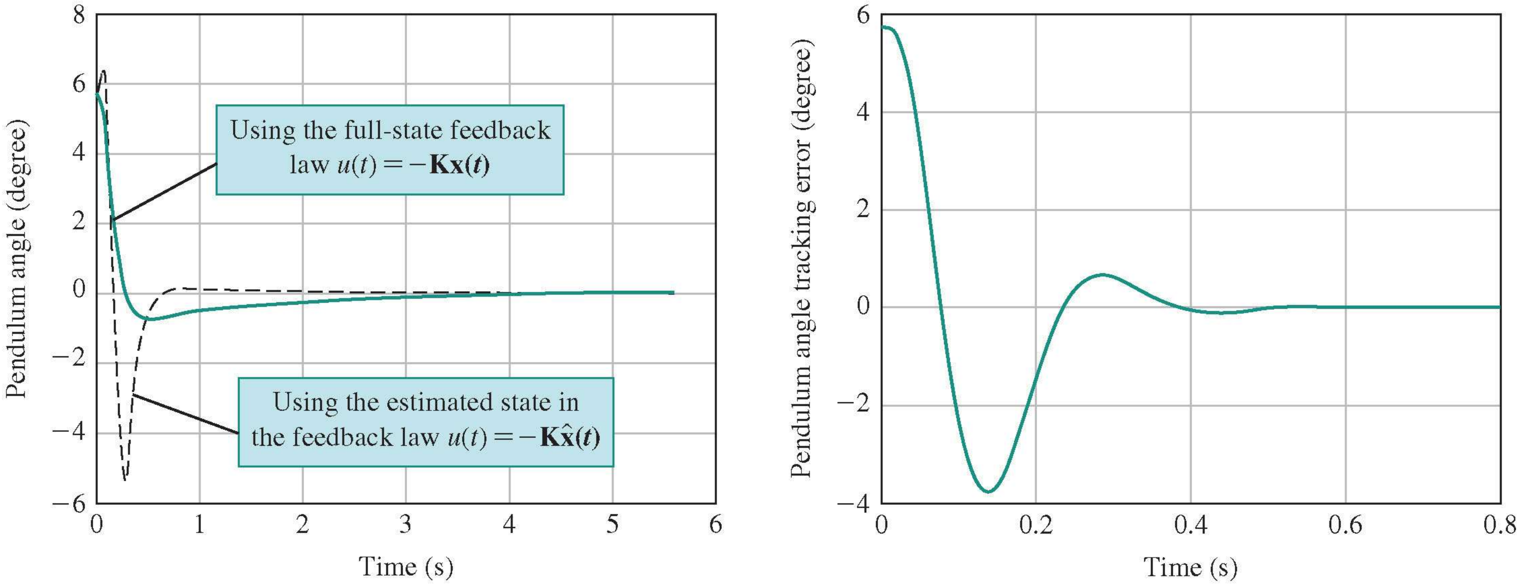
Observer based Feedback Control

Step 3: Compensator Design

The final step in the design is to connect the observer to the full-state feedback control law via

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$$

This stabilize the closed-loop system, however, we should not expect the closed-loop performance to be as good when using the state estimate from the observer.



Observer Design Example

- Given a continuous-time linear time-invariant system, design a Luenberger observer such that the roots of the characteristic equation of the dynamic of the observation error are all located at -3, -3, -4.

$$\dot{x} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

Reference Input and It’s Steady State Error

Reference Input

We referred to the previous design of state variable feedback stabilizing compensators without reference inputs (i.e., r(t) = 0) as ***regulators,*** *however,* ***command following (trajectory tracking)*** *is also an important aspect of feedback design*

Let's consider ***how we can introduce a reference signal*** *into the state variable feedback compensator.*

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) + \mathbf{L}\tilde{y}(t) + \mathbf{M}r(t)$$

$$u(t) = \tilde{u}(t) + Nr(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t),$$

where

$$\tilde{y}(t) = y(t) - \mathbf{C}\hat{\mathbf{x}}(t)$$

The compensator key design parameters required to implement the command tracking of the reference input are ***M and N.***

Recall the 2DoF design for command tracking:

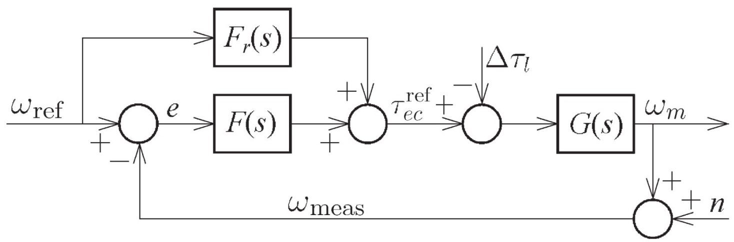


Fig. 4. Block diagram of the 2DOF structure.

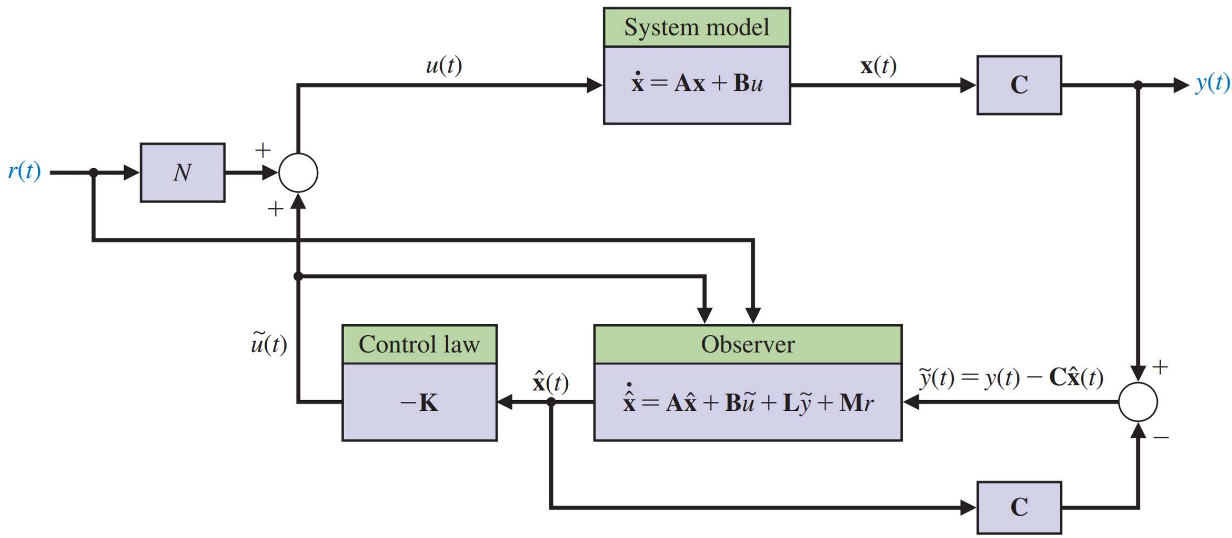


FIGURE 11.11 State variable compensator with a reference input.

Case 1, we select M and N so that the estimation error e(t) is independent of the reference input r(t)

the estimation error

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}\tilde{u}(t) - \mathbf{L}\tilde{y}(t) - \mathbf{M}r(t),$$
$$= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t) + (\mathbf{B}N - \mathbf{M})r(t).$$

Suppose that we select

$$\mathbf{M} = \mathbf{B}N.$$

Then the corresponding estimation error is given by

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).$$

In this case, the estimation error is independent of the reference input, and the remaining task is to determine a suitable value of N.

With M = BN, we find that the compensator is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\tilde{y}(t)$$
$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t).$$

Reference Input

Case 1, we select M and N so that the estimation error e(t) is independent of the reference input r(t)

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\tilde{y}(t)$$
$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t).$$

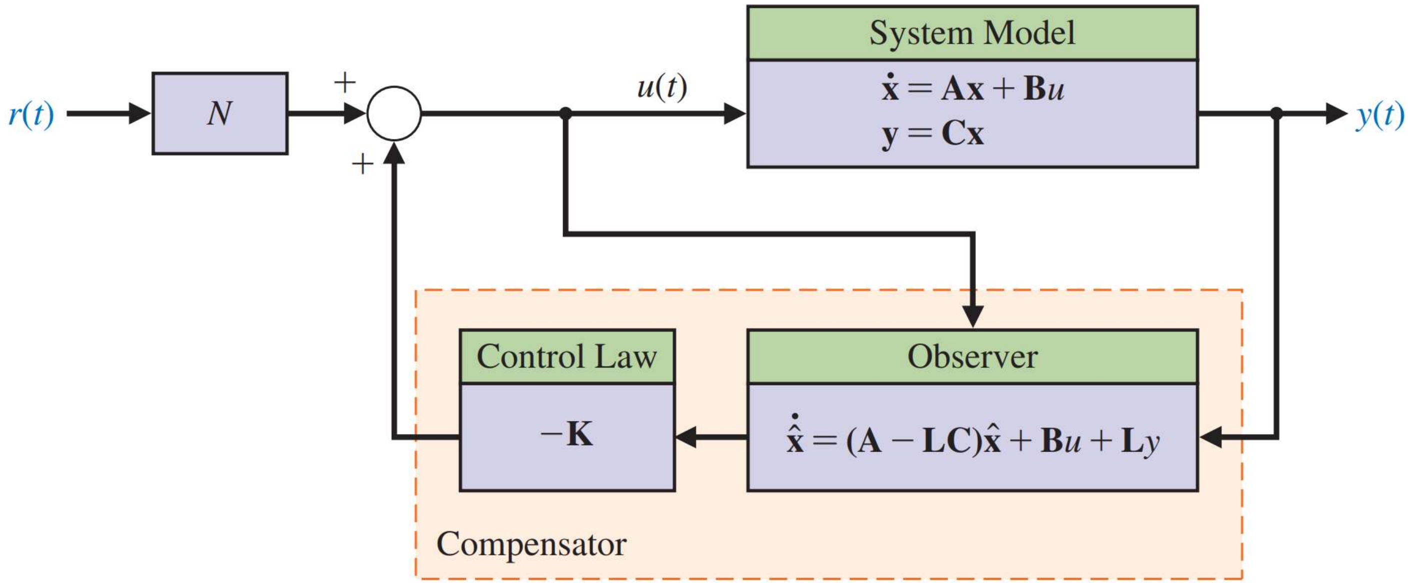


FIGURE 11.12 State variable compensator with reference input and $\mathbf{M} = \mathbf{B}N$.

the compensator is in the feedback loop

Case 2, we select M and N so that the tracking error $y(t) - r(t)$ is used as an input to the compensator.
Recall,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) + \mathbf{L}\tilde{y}(t) + \mathbf{M}r(t)$$

$$u(t) = \tilde{u}(t) + Nr(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t),$$

suppose that we select

$$N = 0 \text{ and } \mathbf{M} = -\mathbf{L}$$

Then, the compensator is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\tilde{y}(t) - \mathbf{L}r(t)$$
$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t),$$

leads to

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e + Lr$$

$$\dot{x} = (A - BK)x + BKe$$

Reference Input

Case 2, we select M and N so that the tracking error y(t) – r(t) is used as an input to the compensator. rewrite the compensator as

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}}(t) + \mathbf{L}(y(t) - r(t))$$
$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t).$$

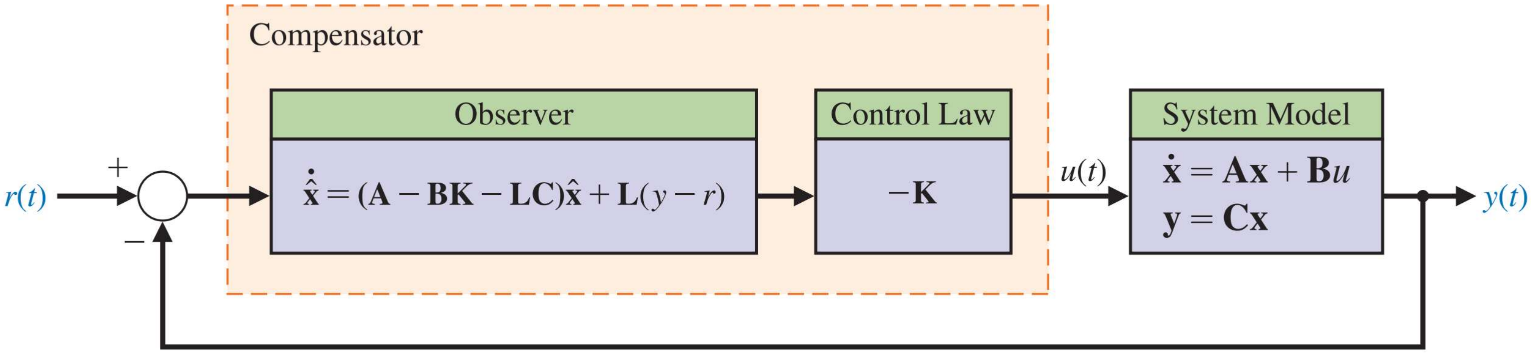


FIGURE 11.13 State variable compensator with reference input and $N = 0$ and $\mathbf{M} = -\mathbf{L}$.

the compensator is in the forward path.

Depending on the choice of N and M, other implementations are possible, for instance, the internal model design.

Internal Model Design

*Now, we consider the problem of designing a compensator that provides asymptotic tracking of **a reference input** with zero steady-state error.*

include steps, ramps, and other persistent signals, such as sinusoids

achieved by type-one system, type-two system, and ??

*This idea is formalized here by **introducing an internal model of the reference input in the compensator***

Consider

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad y(t) = \mathbf{C}\mathbf{x}(t).$$

We consider a reference input to be generated by a linear system of the form

$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t), \qquad r(t) = \mathbf{d}_r\mathbf{x}_r(t),$$

with unknown initial conditions.

For instance, for a step reference input

$$\dot{x}_r(t) = 0, \qquad r(t) = x_r(t),$$

Internal Model Design

Then, the tracking error e(t) is defined as

$$e(t) = y(t) - r(t).$$

Taking the time derivative yields

$$\dot{e}(t) = \dot{y}(t) = \mathbf{C}\dot{\mathbf{x}}(t).$$

If we define the two intermediate variables

$$\mathbf{z}(t) = \dot{\mathbf{x}}(t) \quad \text{and} \quad w(t) = \dot{u}(t),$$

we have

$$\begin{pmatrix} \dot{e}(t) \\ \dot{\mathbf{z}}(t) \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{C} \\ 0 & \mathbf{A} \end{bmatrix} \begin{pmatrix} e(t) \\ \mathbf{z}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix} w(t).$$

If the system is controllable, we can find a feedback of the form

$$w(t) = -K_1 e(t) - \mathbf{K}_2 \mathbf{z}(t)$$

such that the system is stable.

This implies we will have achieved the objective of asymptotic tracking with zero steady state error.

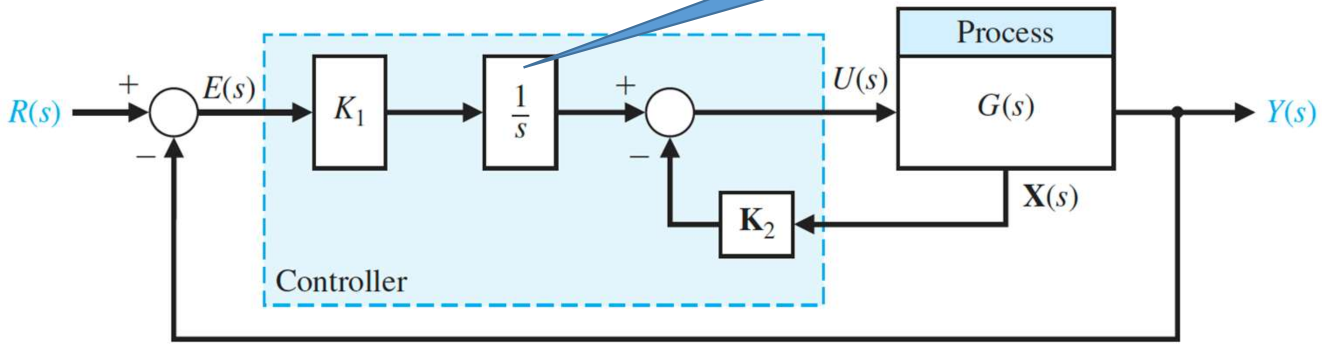
Internal Model Design

The control input, found by

$$u(t) = -K_1 \int_0^t e(\tau)\, d\tau - \mathbf{K}_2 \mathbf{x}(t).$$

The corresponding block diagram

The internal Model



The internal model principle states that if G(s)Gc(s) contains R(s), then y(t) will track r(t) asymptotically.

achieved by type-one system, type-two system, and controller contains $\frac{\omega}{s^2+\omega^2}$

Internal Model Design

- Example: consider the following state space model, design a state feedback controller $u = u(x)$, that tracks a step reference input $r(s) = \frac{1}{s}$ with zero steady state error $e = y - r$.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(t).$$

$$u(t) = -K_1 \int_0^t e(\tau)\, d\tau - \mathbf{K}_2\mathbf{x}(t).$$

- Further design the controller with output feedback control $u = u(y)$ and the input is a ramp $r = \frac{1}{s^2}$.

Internal Model Design

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- Design an internal model controller for a dc motor (3rd order model) with its angular position tracking a ramp input reference signal $r = \frac{1}{s^2}$.

1. 核心思路：重新定义误差

为了把那个“讨厌的” -1 变成 $+1$，我们只需将误差定义反转：

$$e(t) \triangleq y(t) - r(t) = \Theta(t) - r(t)$$

2. 构建“积分器链” (Chain of Integrators)

为了构造能控规范型，我们需要证明系统状态之间存在层层递进的导数关系。让我们按照**从左到右**（从内模最深处到控制输入端）的顺序梳理变量：

利用简化参数 ($K_T = 1, \ldots$)，系统的物理链条是：

1. **电流** i 驱动 **角加速度** $\dot{\Omega}$ ($\dot{\Omega} = i$)
2. **角速度** Ω 驱动 **角速度** $\dot{\Theta}$ ($\dot{\Theta} = \Omega$)
3. **角度** Θ 驱动 **误差积分** ($\dot{z}_2 = e = \Theta - r$)
4. **误差积分** z_2 驱动 **双重积分** z_1 ($\dot{z}_1 = z_2$)

这形成了一个完美的单向链条：

$$z_1 \xleftarrow{\int} z_2 \xleftarrow{\int} \Theta \xleftarrow{\int} \Omega \xleftarrow{\int} i \leftarrow u$$

逐行写出微分方程：

1. **第一行（内模）：** $\dot{z}_1 = z_2$
2. **第二行（耦合）：** $\dot{z}_2 = e = \Theta - r$ *(注意：这里 Θ 的系数现在是 +1 了)*
3. **第三行（运动学）：** $\dot{\Theta} = \Omega$
4. **第四行（动力学）：** $\dot{\Omega} = i$ *(简化参数下 $J = 1, B_d = 0$)*
5. **第五行（电气）：** $\dot{i} = -\Omega - i + u$ *(简化参数下 $L = 1, R = 1, K_E = 1$)*

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \Theta \\ \Omega \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}}_{A_{aug}} \begin{bmatrix} z_1 \\ z_2 \\ \Theta \\ \Omega \\ i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B_{aug}} u + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{E_{aug}} r$$

$$u = -Kx_a = -\begin{bmatrix} k_{z1} & k_{z2} & k_\theta & k_\omega & k_i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \Theta \\ \Omega \\ i \end{bmatrix}$$

- $z_2(t) = \int_0^t (\Theta - r) d\tau = \int_0^t e(\tau) d\tau$
- $z_1(t) = \int_0^t z_2(\tau) d\tau = \int_0^t \int_0^\sigma e(\tau) d\tau d\sigma$

Optimal Control (LQR)

Optimal Control Design

Definition: The design of a systems that are adjusted to provide a minimum performance index such as

$$J = \int_0^{\infty} g(\mathbf{x}, \mathbf{u}, t)\, dt,$$

are called optimal control systems

Consider the LTI SISO system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).$$

select a feedback controller as

$$u(t) = -\mathbf{K}\mathbf{x}(t),$$

yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = \mathbf{H}\mathbf{x}(t),$$

where H is the n x n matrix.

Optimal Control Design

Consider an error-squared performance index

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t)\,dt.$$

where

$$\mathbf{x}^T(t)\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), \ldots, x_n(t))\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$= x_1^{\,2}(t) + x_2^{\,2}(t) + x_3^{\,2}(t) + \cdots + x_n^{\,2}(t),$$

To obtain the minimum value of J, we postulate the existence of an exact differential so that

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)) = -\mathbf{x}^T(t)\mathbf{x}(t),$$

where P is to be determined. A symmetric P matrix will be used to simplify the algebra without any loss of generality. The scalar function $x^T Px$ is known as Lyapunov function.

Differentiating the Lyapunov function $V = x^T P x$ gives

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)) = \dot{\mathbf{x}}^T(t)\mathbf{P}\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{P}\dot{\mathbf{x}}(t).$$

$$= \mathbf{x}^T(t)(\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H})\mathbf{x}(t).$$

If we let

$$\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H} = -\mathbf{I},$$

If H is Hurwitz, the existence of an symmetric and positive definite matrix P is guaranteed, this equation is aka the **Lyapunov equation**

then

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)) = -\mathbf{x}^T(t)\mathbf{x}(t),$$

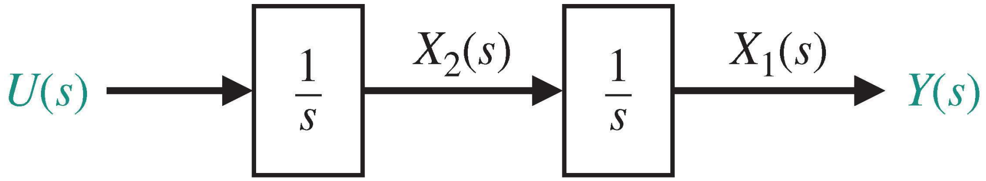
which indicates to

$$J = \int_0^\infty -\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t))\,dt = -\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)\Big|_0^\infty = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0).$$

The design steps are then as follows

1. Determine the matrix P that satisfies above Lyapunov equation, where H is known.
2. Minimize J by determining the minimum of $x^T(0)Px(0)$ *by adjusting one or more unspecified system parameters.*

Example:



The vector differential equation of this system is

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

We choose a feedback control system so that

$$u(t) = -k_1 x_1(t) - k_2 x_2(t),$$

Then the system becomes

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -k_1 x_1(t) - k_2 x_2(t).$$

Example:

In matrix form, we have

$$\dot{\mathbf{x}}(t) = \mathbf{H}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\mathbf{x}(t).$$

Let $k_1 = 1$ *and determine a suitable value for* k_2 *so that the performance index is minimized.*

From the Lyapunov equation, it follows that

$$\begin{bmatrix} 0 & -1 \\ 1 & -k_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Completing the matrix multiplication and addition yields

$$\begin{aligned} -p_{12} - p_{12} &= -1, \\ p_{11} - k_2 p_{12} - p_{22} &= 0, \\ p_{12} - k_2 p_{22} + p_{12} - k_2 p_{22} &= -1. \end{aligned}$$

$$\Rightarrow \quad p_{12} = \frac{1}{2}, \qquad p_{22} = \frac{1}{k_2}, \qquad p_{11} = \frac{{k_2}^2 + 2}{2k_2}.$$

Example:

Consider the integral performance index is then

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0),$$

where

$$\mathbf{x}^T(0) = (1, 1).$$

Therefore J becomes

$$J = [1 \quad 1]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = p_{11} + 2p_{12} + p_{22}.$$

Substituting the values of the elements of P, we have

$$J = \frac{k_2{}^2 + 2}{2k_2} + 1 + \frac{1}{k_2} = \frac{k_2{}^2 + 2k_2 + 4}{2k_2}.$$

To minimize as a function of k2,

$$\frac{dJ}{dk_2} = \frac{2k_2(2k_2 + 2) - 2(k_2{}^2 + 2k_2 + 4)}{(2k_2)^2} = 0.$$

Example:

Therefore

$$k_2 = 2$$

when J is a minimum. The minimum value of J is

$$J_{\min} = 3.$$

The system matrix H, obtained for the compensated system, is then

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

The characteristic equation of the compensated system is therefore

$$\det[\lambda \mathbf{I} - \mathbf{H}] = \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix} = \lambda^2 + 2\lambda + 1.$$

therefore the damping ratio of the compensated system is $\xi = 1$

we recognize that this system is optimal only for the specific set of initial conditions that were assumed.

Example continue:

let us consider again

$$\dot{\mathbf{x}}(t) = \mathbf{H}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\mathbf{x}(t).$$

with $k_1 = k_2 = k$. *Then system becomes*

$$\dot{\mathbf{x}}(t) = \mathbf{H}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix}\mathbf{x}(t).$$

To determine the P matrix, we use the Lyapunov equation, yielding

$$p_{12} = \frac{1}{2k}, \qquad p_{22} = \frac{k+1}{2k^2}, \quad \text{and} \quad p_{11} = \frac{1+2k}{2k}.$$

Let us consider the case

$$\mathbf{x}^T(0) = (1 \quad 0)$$

Example continue:

Then the performance index becomes

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t)\,dt = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) = p_{11} = \frac{1+2k}{2k} = 1 + \frac{1}{2k}.$$

Then the minimum value of J is obtained when k approaches infinity, which is 1.

Now, we recognize that, in providing a very large gain k, we can cause the feedback signal

$$u(t) = -k(x_1(t) + x_2(t))$$

to be very large, which is unrealistic, cause in many cases, we have physical limits on the control magnitude.

We can limit the control effort by including it within the expression for the performance index

$$J = \int_0^\infty (\mathbf{x}^T(t)\mathbf{I}\mathbf{x}(t) + \lambda\mathbf{u}^T(t)\mathbf{u}(t))\,dt,$$

The weighting factor λ will be chosen so that the relative importance of the state variable performance is contrasted with the importance of the control energy.

Optimal Control Design

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Example continue:

Now, let us consider again when λ *is other than zero and account for the expenditure of control signal energy.*

we still use a state variable feedback

$$u(t) = -\mathbf{K}\mathbf{x}(t) = [-k \quad -k]\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The performance function becomes

$$J = \int_0^\infty \mathbf{x}^T(t)(\mathbf{I} + \lambda\mathbf{K}^T\mathbf{K})\mathbf{x}(t)\,dt = \int_0^\infty \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)\,dt,$$

$$\mathbf{Q} = \mathbf{I} + \lambda\mathbf{K}^T\mathbf{K} = \begin{bmatrix} 1 + \lambda k^2 & \lambda k^2 \\ \lambda k^2 & 1 + \lambda k^2 \end{bmatrix}.$$

let $\mathbf{x}^T(0) = (1, \quad 0)$ *yielding*

$$J = p_{11} = (1 + \lambda k^2)\left(1 + \frac{1}{2k}\right) - \lambda k^2. \quad \Rightarrow \quad \frac{dJ}{dk} = \frac{1}{2}\left(\lambda - \frac{1}{k^2}\right) = 0.$$

Therefore, the minimum of the performance index occurs when

$$k = k_{\min} = 1/\sqrt{\lambda},$$

Optimal Control Design

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Linear Quadratic Regulator (LQR)

Previous design procedure can be carried out for a more general LTI SISO systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

with feedback

$$u(t) = -\mathbf{K}\mathbf{x}(t) = -[k_1 \quad k_2 \ldots k_n]\mathbf{x}(t).$$

We can consider the performance index

$$J = \int_0^\infty (\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + Ru^2(t))\, dt,$$

where $R>0$ is a scalar weighting factor. This index is minimized when

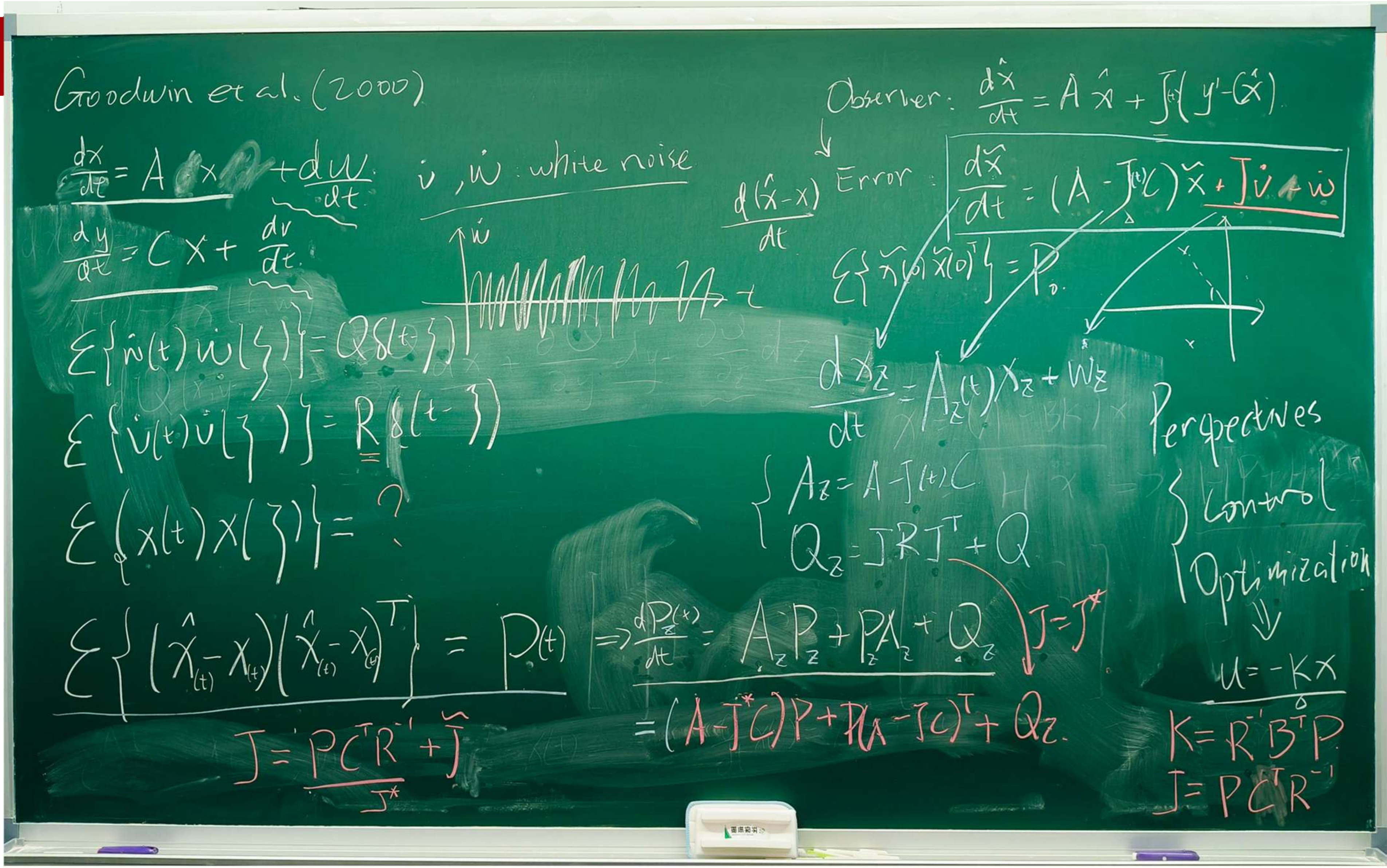
$$\mathbf{K} = R^{-1}\mathbf{B}^T\mathbf{P}.$$

The $n \times n$ matrix P is determined from the solution of the equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

which is often called the algebraic Riccati equation.

Kalman Filter (Optional)


Goodwin et al. (2000)
dx/dt = A x + dw/dt
v, w : white noise
dy/dt = C x + dv/dt
Observer: dx̂/dt = A x̂ + J(t)(y' - Cx̂)
Error: dx̃/dt = (A - J(t)C) x̃ + J v + w
d(x̂ - x)/dt
E{x̃(0) x̃(0)ᵀ} = P₀
E{w(t) w(ξ)} = Q δ(t - ξ)
E{v(t) v(ξ)} = R δ(t - ξ)
E{x(t) x(ξ)} = ?
dx_z/dt = A_z(t) x_z + w_z
A_z = A - J(t) C
Q_z = J R Jᵀ + Q
Perspectives
Control
Optimization
u = -Kx
K = R⁻¹ Bᵀ P
J = P Cᵀ R⁻¹
E{(x̂(t) - x(t))(x̂(t) - x(t))ᵀ} = P(t)
dP_z(t)/dt = A_z P_z + P_z A_zᵀ + Q_z
= (A - J*C) P + P(A - JC)ᵀ + Q_z
J = J*
J = P Cᵀ R⁻¹ + J̃
J*

Summary

Regulator problem

Controllability
Full-state feedback control law
Pole placement
Separation principle
Observability
Observer
Estimation error

Command following
Internal model design

Linear quadratic regulator

Output Regulation

Robust control

Kalman state-space decomposition

Kalman filter
Reduced-order observer design

Adaptive control

Optimal control
MPC

Kalman Decomposition

- Kalman Decomposition:
  - Controllable and Observable
  - Uncontrollable and Unobservable
  - Controllable and Unobservable
  - Uncontrollable and Observable