上海科技大学
ShanghaiTech University

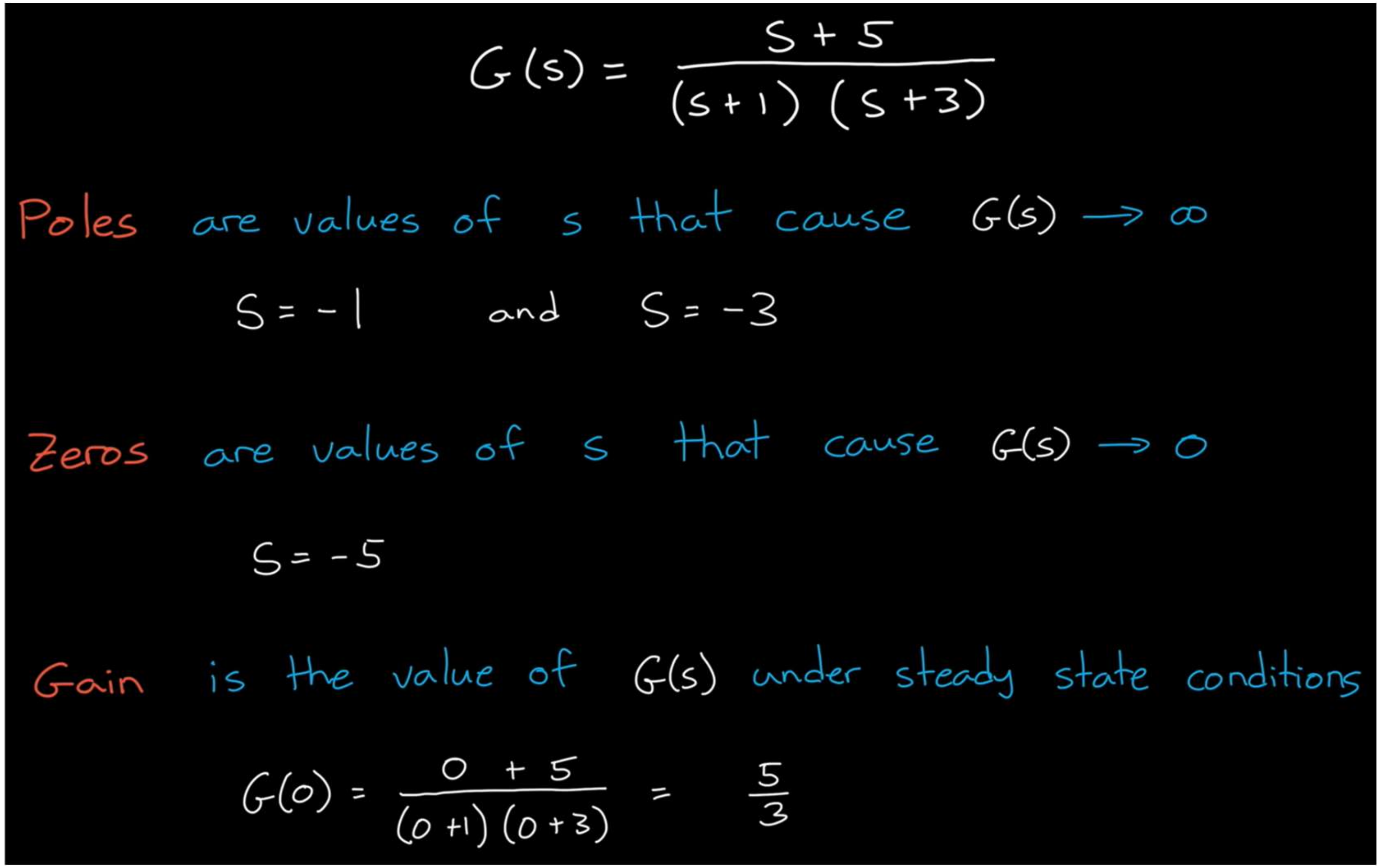
Lecture 3: Stability

Chen, Jiahao
SIST 1D#206

Poles and Zeros

Poles and Zeros

- Pole, zero, and gain defined by Douglas:



Poles and Zeros

- Pole due to input -> forced response
- Pole due to transfer function -> natural response
- Zero and pole both contribute to the amplitudes of the response
- Recall the solution of the state space model:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$= \boldsymbol{\Phi}(t)\mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

zero-input response (natural response)

Zero-state response (forced response) as convolution integral

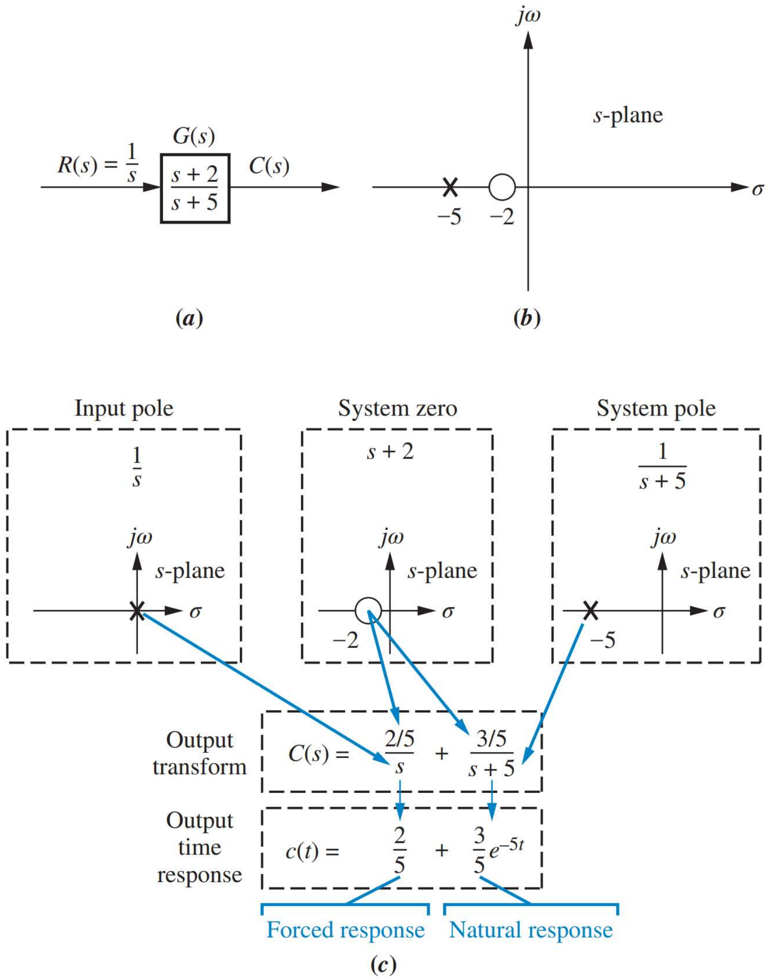


FIGURE 4.1 **a.** System showing input and output; **b.** pole-zero plot of the system; **c.** evolution of a system response. Follow blue arrows to see the evolution of the response component generated by the pole or zero.

Evaluating Response Using Poles

PROBLEM: Given the system of Figure 4.3, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}} \tag{4.3}$$

Taking the inverse Laplace transform, we get

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}} \tag{4.4}$$

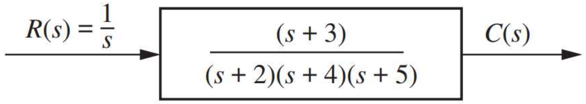


FIGURE 4.3 System for Example 4.1

PROBLEM: A system has a transfer function, $G(s) = \frac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$. Write, by inspection, the output, $c(t)$, in general terms if the input is a unit step.

ANSWER: $c(t) \equiv A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$

Typical Systems
(excited with test signal)

First order system

- Step response of the first order system (with no zero)
  - Forced and natural response:

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

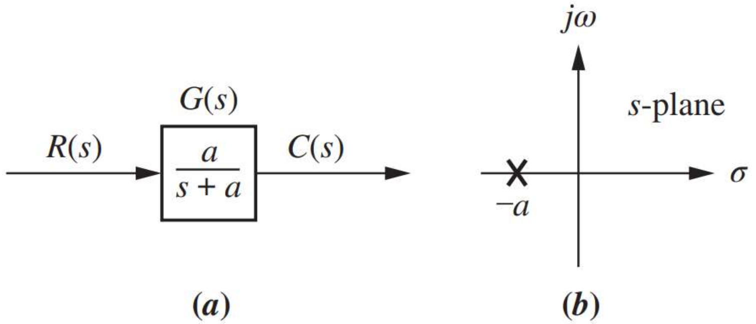


FIGURE 4.4 **a.** First-order system; **b.** pole plot

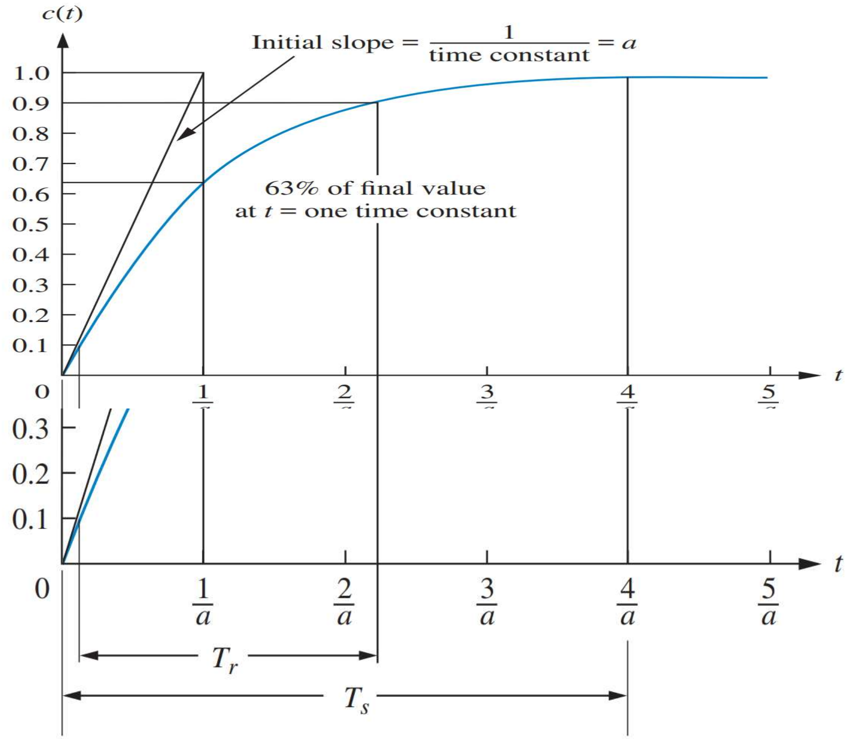


FIGURE 4.5 First-order system response to a unit step

Second order system

- Step response of the second order system (with no zero) depends on pole locations on the pzmap:
  - How about right plane pole?

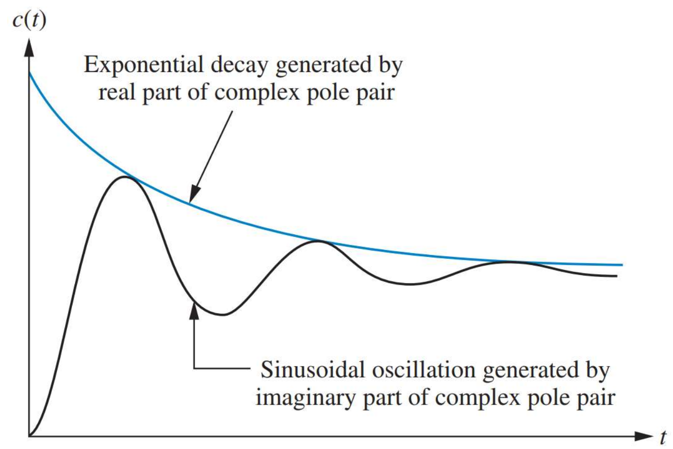


FIGURE 4.8 Second-order step response components generated by complex poles

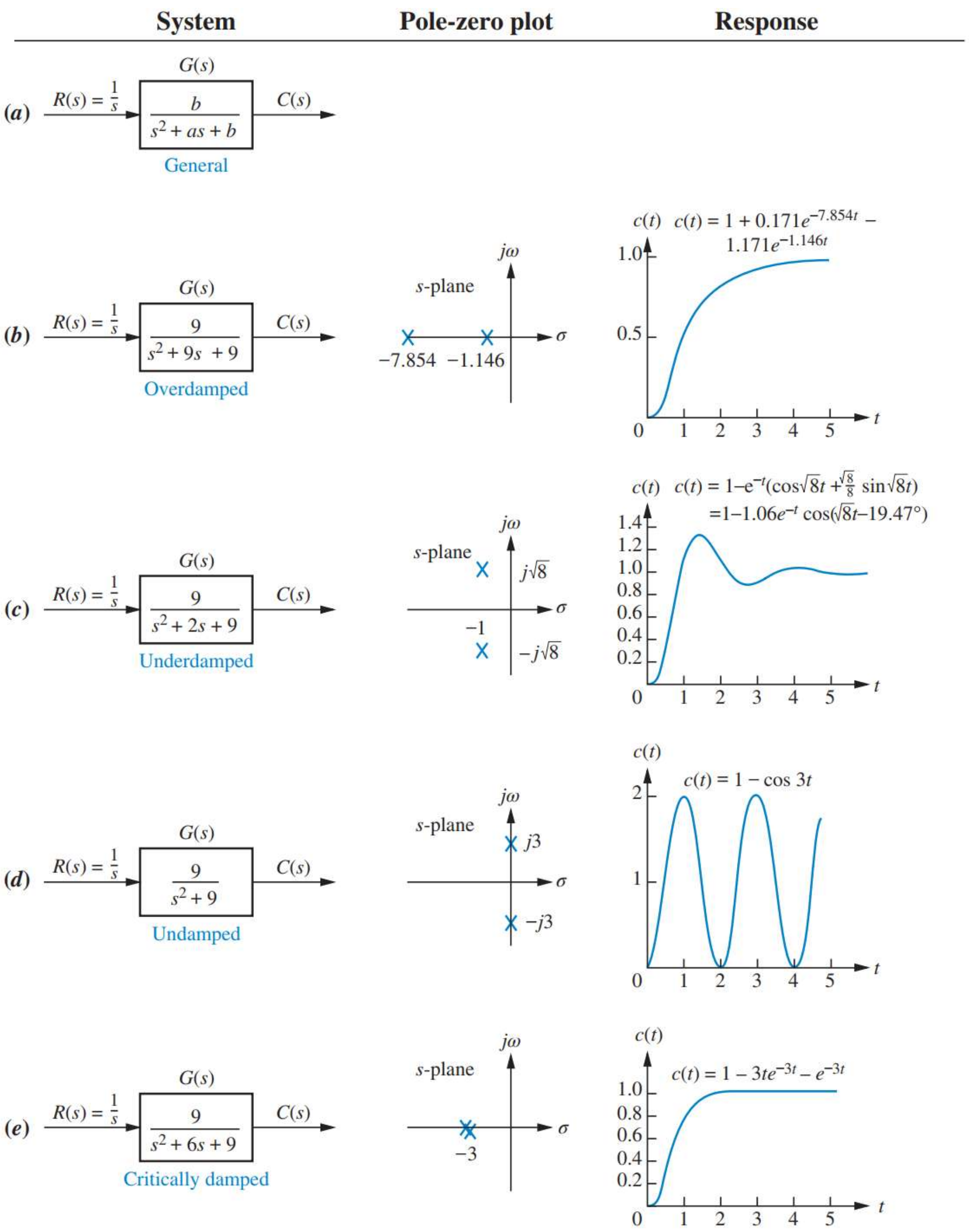


FIGURE 4.7 Second-order systems, pole plots, and step responses

Second order system

上海科技大学 ShanghaiTech University

- Natural frequency ω_n defines the time scale of the time response plot
  - In s-plane, it defines the length of the pole location vector
- Damping ratio ζ defines the shape of the time response plot
  - In s-plane, it defines the angle of the pole location vector

$$\boxed{G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

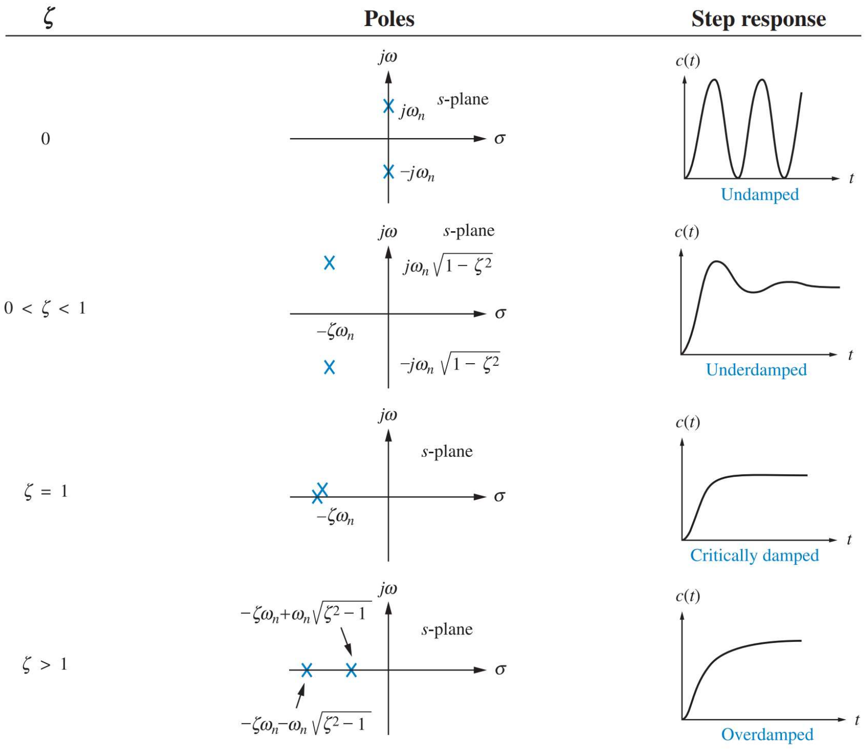


FIGURE 4.11 Second-order response as a function of damping ratio

Finding ζ and ω_n For a Second-Order System

PROBLEM: Given the transfer function of Eq. (4.23), find ζ and ω_n.

$$G(s) = \frac{36}{s^2 + 4.2s + 36} \tag{4.23}$$

SOLUTION: Comparing Eq. (4.23) to (4.22), $\omega_n^2 = 36$, from which $\omega_n = 6$. Also, $2\zeta\omega_n = 4.2$. Substituting the value of ω_n, $\zeta = 0.35$.

Time domain performance metrics in time domain

- Time domain performance metrics are defined in terms of the underdamped response of a second order system with $\zeta \in (0,1)$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$\begin{aligned} c(t) &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi) \end{aligned}$$

where $\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$.

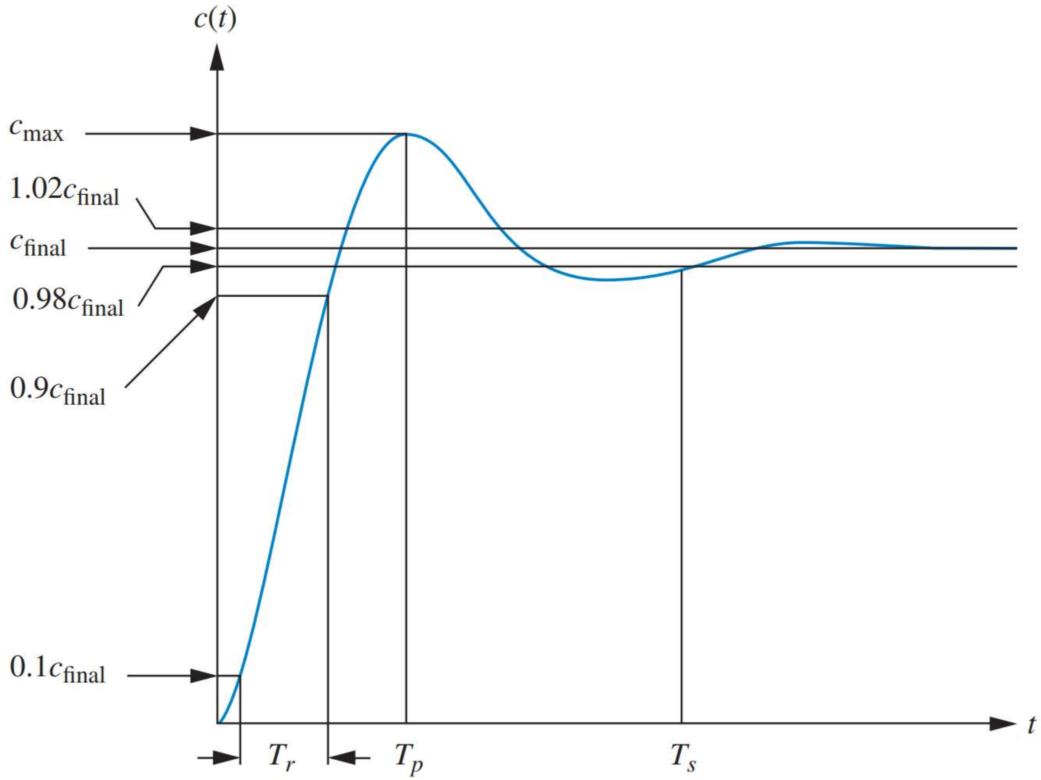


FIGURE 4.14 Second-order underdamped response specifications

Time domain performance metrics in time domain

Finding T_p, %OS, T_s, and T_r from a Transfer Function

PROBLEM: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100} \qquad (4.43)$$

find T_p, $\%OS$, T_s, and T_r.

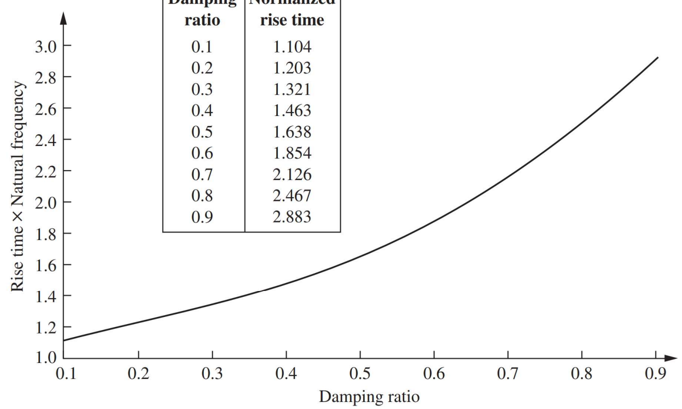
SOLUTION: ω_n and ζ are calculated as 10 and 0.75, respectively. Now substitute ζ and ω_n into Eqs. (4.34), (4.38), and (4.42) and find, respectively, that $T_p = 0.475$ second, $\%OS = 2.838$, and $T_s = 0.533$ second. Using the table in Figure 4.16, the

SOLUTION: ω_n and ζ are calculated as 10 and 0.75, respectively. Now substitute ζ and ω_n into Eqs. (4.34), (4.38), and (4.42) and find, respectively, that $T_p = 0.475$ second, $\%OS = 2.838$, and $T_s = 0.533$ second. Using the table in Figure 4.16, the normalized rise time is approximately 2.3 seconds. Dividing by ω_n yields $T_r = 0.23$ second. This problem demonstrates that we can find T_p, $\%OS$, T_s, and T_r without the tedious task of taking an inverse Laplace transform, plotting the output response, and taking measurements from the plot.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad (4.34)$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 \qquad (4.38)$$

$$T_s = \frac{4}{\zeta\omega_n} \qquad (4.42)$$



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

FIGURE 4.16 Normalized rise time versus damping ratio for a second-order underdamped response

Percent overshoot, %OS. The amount that the waveform overshoots the steady-state value (or final value) at the peak time, expressed as a percentage of the steady state value.

Time domain performance metrics in s-domain

- Peak time and settling time are related to :
  - damped frequency of oscillation, ω_d
  - and exponential damping frequency σ_d

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

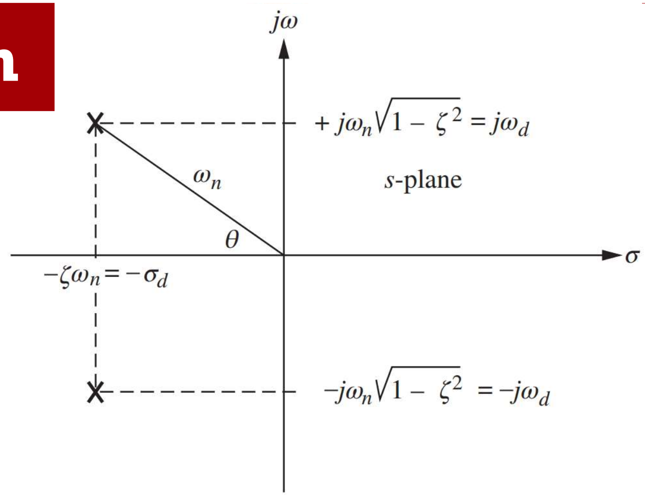


FIGURE 4.17 Pole plot for an underdamped second-order system

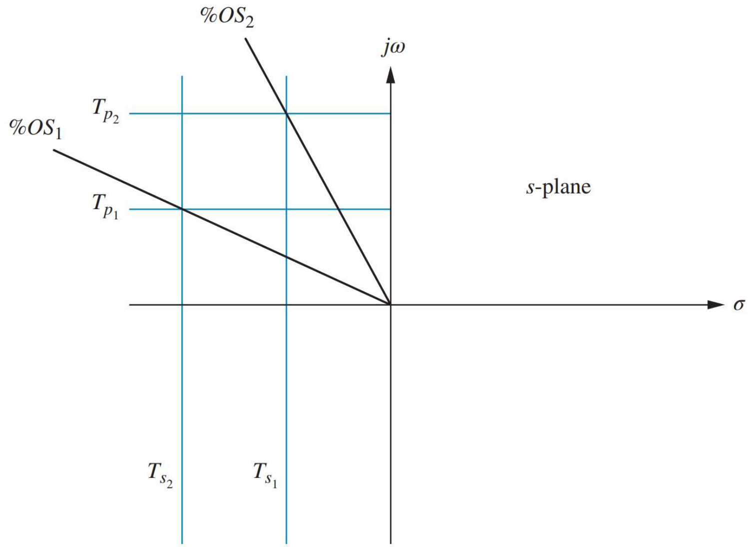


FIGURE 4.18 Lines of constant peak time, T_p, settling time, T_s, and percent overshoot, $\%OS$. Note: $T_{s_2} < T_{s_1}$; $T_{p2} < T_{p1}$; $\%OS_1 < \%OS_2$.

Time domain performance metrics in s-domain

- Example responses

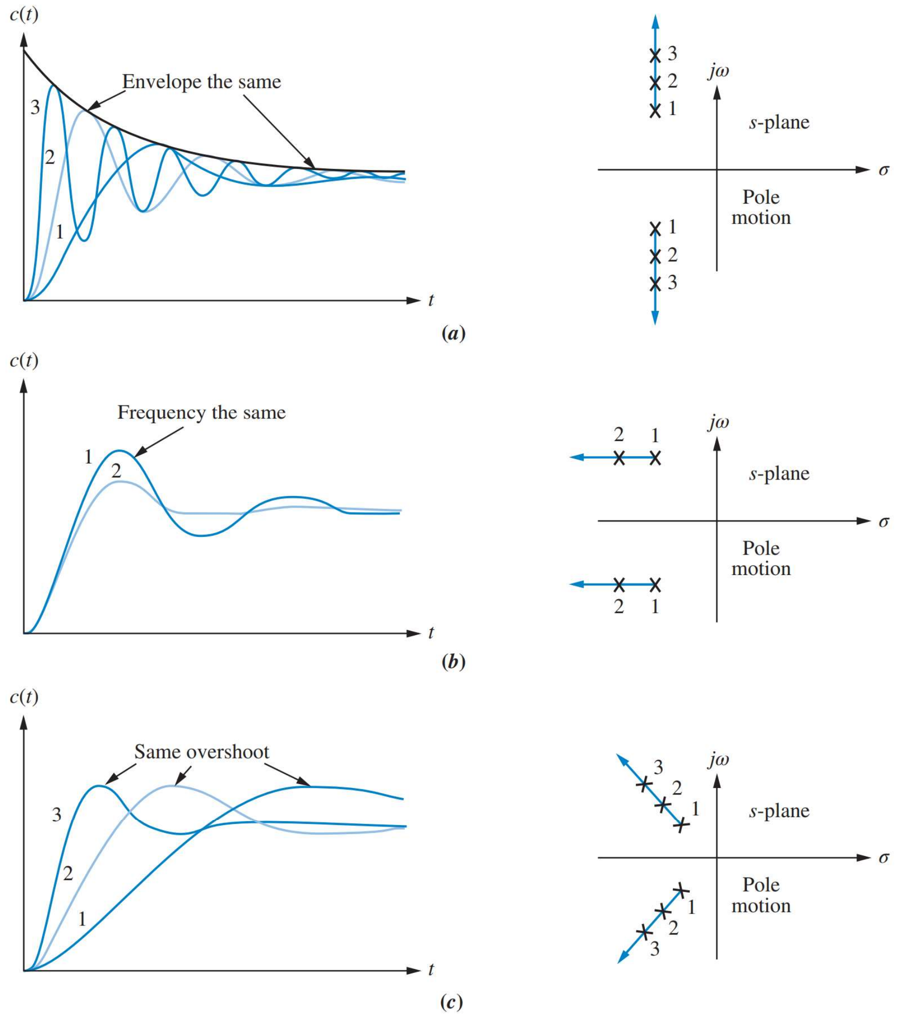


FIGURE 4.19 Step responses of second-order underdamped systems as poles move: **a.** with constant real part; **b.** with constant imaginary part; **c.** with constant damping ratio

上海科技大学 ShanghaiTech University

Finding T_p, %OS, and T_s from Pole Location

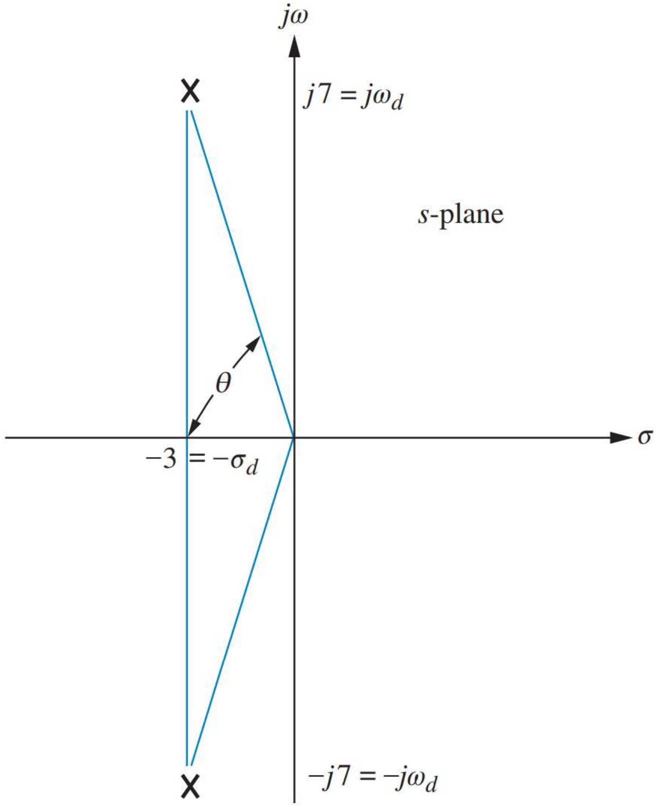


FIGURE 4.20 Pole plot for Example 4.6

PROBLEM: Given the pole plot shown in Figure 4.20, find ζ, ω_n, T_p, $\%OS$, and T_s.

SOLUTION: The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second} \tag{4.46}$$

The percent overshoot is

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\% \tag{4.47}$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds} \tag{4.48}$$

Stability

(in terms of total response due to bounded input)

Bounded input bounded output (BIBO) stability

- A system is **stable** if every bounded input yields a bounded output.
- A system is **unstable** if any bounded input yields an unbounded output.
- **Marginal stability** means that the system is stable for some bounded inputs and unstable for others.
  - Poles of multiplicity greater than 1 on the imaginary axis lead to the sum of responses of the form $At^n\cos(\omega t + \phi)$, where $n = 1,2,3 \ldots$ where the amplitude approaches infinity as time approaches infinity.

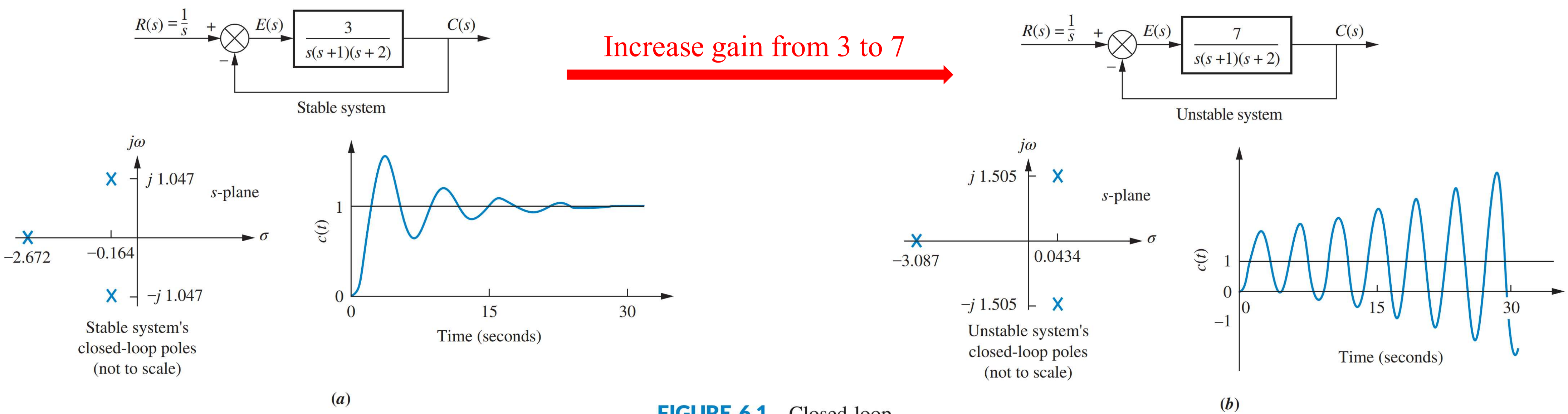


FIGURE 6.1 Closed-loop poles and response: **a.** stable system; **b.** unstable system

Stability

(in terms of Routh-Hurwitz Criterion)

Routh Table

- Not easy to find **closed loop poles** given **open loop poles**, e.g.

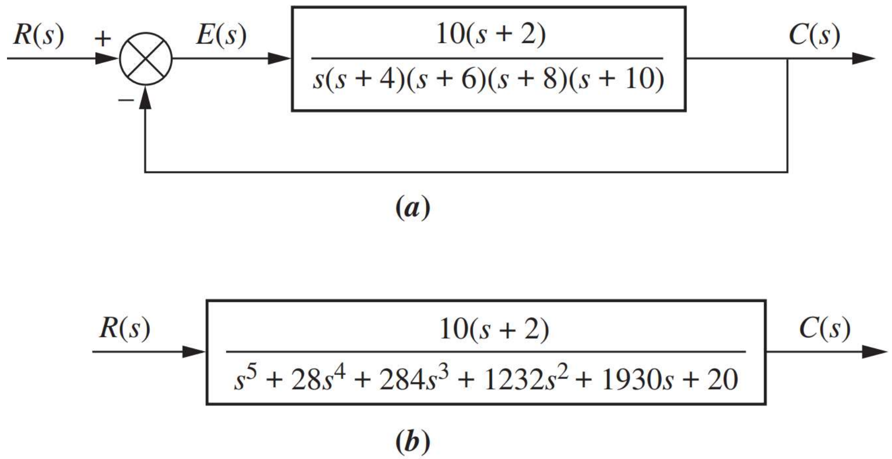


FIGURE 6.2 Common cause of problems in finding closed-loop poles: **a.** original system; **b.** equivalent system

- We can quickly learn the number of poles in the right half s-plane and $j\omega$-axis by writing down a table of coefficients, called Routh table.

TABLE 6.1 Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Routh Table

- Basic Routh table example

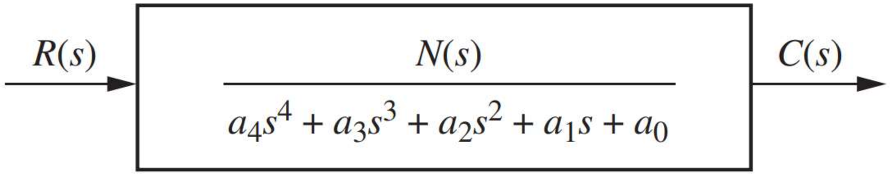


FIGURE 6.3 Equivalent closed-loop transfer function

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Creating a Routh Table

PROBLEM: Make the Routh table for the system shown in Figure 6.4(*a*).

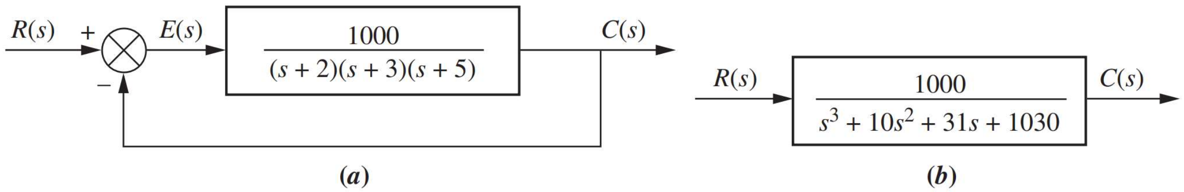


FIGURE 6.4 **a.** Feedback system for Example 6.1; **b.** equivalent closed-loop system

Routh-Hurwitz Criterion

- Simply stated, the **Routh-Hurwitz criterion** declares that the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column of Routh table

PROBLEM: Make a Routh table and tell how many roots of the following polynomial are in the right half-plane and in the left half-plane.

$$P(s) = 3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

ANSWER:

https://routhhurwitz.streamlit.app/

Routh-Hurwitz Table Generator

Polynomial

Enter polynomial coefficients separated by commas

6,2,8,7,4,6,9,3

$$3s^7 + 9s^6 + 6s^5 + 4s^4 + 7s^3 + 8s^2 + 2s + 6$$

Go

Routh-Hurwitz Table

s^7	3	6	7	2
s^6	9	4	8	6
s^5	$\frac{14}{3}$	$\frac{13}{3}$	0	0
s^4	$-\frac{61}{14}$	8	6	0
s^3	$\frac{787}{61}$	$\frac{392}{61}$	0	0
s^2	$\frac{8004}{787}$	6	0	0
s	$-\frac{1581}{1334}$	0	0	0
1	6	0	0	0

Routh Table with Zero Only in the First Column (1)

上海科技大学 ShanghaiTech University

- If the first element of a row is zero, division by zero would be required to form the next row. To avoid this phenomenon, an epsilon, ϵ, is assigned to replace the zero in the first column. The value ϵ is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

---known as **epsilon method.**

TryIt 6.1

Use the following MATLAB statement to find the poles of the closed-loop transfer function in Eq. (6.1).

```
roots([1 2 3 6 5 3])
```

Stability via Epsilon Method

PROBLEM: Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \tag{6.2}$$

SOLUTION: The solution is shown in Table 6.4. We form the Routh table by using the denominator of Eq. (6.2). Begin by assembling the Routh table down to the row where a zero appears *only* in the first column (the s^3 row). Next replace the zero by a small number, ϵ, and complete the table. To begin the interpretation, we must first assume a sign, positive or negative, for the quantity ϵ. Table 6.5 shows the first column of Table 6.4 along with the resulting signs for choices of ϵ positive and ϵ negative.

TABLE 6.4 Completed Routh table for Example 6.2

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}$ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

TABLE 6.5 Determining signs in first column of a Routh table with zero as first element in a row

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	$\cancel{0}$ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Routh Table with Zero Only in the First Column (2)

- Alternatively, there is a reverse coefficient method that is suited to be numerically implemented.
  - It is valid because we only care about the location of the poles (left or right half plane)
  - Let $s = 1/d$

$$\left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \cdots + a_1\left(\frac{1}{d}\right) + a_0 = 0$$

```
G =

                 10
  ------------------------------------
  s^5 + 2 s^4 + 3 s^3 + 6 s^2 + 5 s + 3

Continuous-time transfer function.
Model Properties

RH_Table =

    1.0000    3.0000    5.0000         0
    2.0000    6.0000    3.0000         0
         0    3.5000         0         0
      -Inf       NaN       NaN         0
       NaN       NaN       NaN         0
       NaN       NaN       NaN         0

 *You have 0 right-half poles and 5 left-half poles

RH_Table =

    3.0000    6.0000    2.0000         0
    5.0000    3.0000    1.0000         0
    4.2000    1.4000         0         0
    1.3333    1.0000         0         0
   -1.7500         0         0         0
    1.0000         0         0         0

 *Upon changes made to the RH Table, you now have 2
>> roots([1,2,3,6,5,3])

ans =

   0.3429 + 1.5083i
   0.3429 - 1.5083i
  -1.6681 + 0.0000i
  -0.5088 + 0.7020i
  -0.5088 - 0.7020i
```

Stability via Reverse Coefficients

PROBLEM: Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \tag{6.6}$$

SOLUTION: First write a polynomial that has the reciprocal roots of the denominator of Eq. (6.6). From our discussion, this polynomial is formed by writing the denominator of Eq. (6.6) in reverse order. Hence,

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1 \tag{6.7}$$

We form the Routh table as shown in Table 6.6 using Eq. (6.7). Since there are two sign changes, the system is unstable and has two right-half-plane poles. This is the same as the result obtained in Example 6.2. Notice that Table 6.6 does not have a zero in the first column.

TABLE 6.6 Routh table for Example 6.3

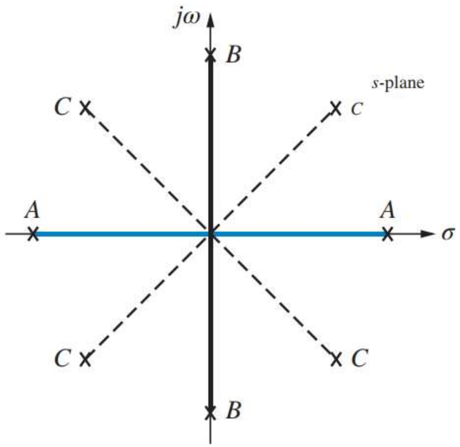
s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

https://jp.mathworks.com/matlabcentral/fileexchange/72884-routh-hurwitz-stability-criteria-table-generator-all-cases

Routh Table with Entire Row being Zero

- When there is an even polynomial that is a factor of the original polynomial, an entire row consists of zeros will result in Routh table. We can write down an auxiliary polynomial and take derivative w.r.t. s to continue with the Routh table

---known as **auxiliary polynomial method.**



A: Real and symmetrical about the origin
B: Imaginary and symmetrical about the origin
C: Quadrantal and symmetrical about the origin

FIGURE 6.5 Root positions to generate even polynomials: A, B, C, or any combination

Stability via Routh Table with Row of Zeros

PROBLEM: Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56} \tag{6.8}$$

SOLUTION: Start by forming the Routh table for the denominator of Eq. (6.8) (see Table 6.7). At the second row we multiply through by 1/7 for convenience. We stop at the third row, since the entire row consists of zeros, and use the following procedure. First we return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients. The polynomial will start with the power of s in the label column and continue by skipping every other power of s. Thus, the polynomial formed for this example is

$$P(s) = s^4 + 6s^2 + 8 \tag{6.9}$$

Next we differentiate the polynomial with respect to s and obtain

$$\frac{dP(s)}{ds} = 4s^3 + 12s + 0 \tag{6.10}$$

Finally, we use the coefficients of Eq. (6.10) to replace the row of zeros. Again, for convenience, the third row is multiplied by 1/4 after replacing the zeros.

The remainder of the table is formed in a straightforward manner by following the standard form shown in Table 6.2. Table 6.7 shows that all entries in the first column are positive. Hence, there are no right-half-plane poles.

TABLE 6.7 Routh table for Example 6.4

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ ~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Routh Table with Entire Row being Zero

PROBLEM: Use the Routh-Hurwitz criterion to find how many poles of the following closed-loop system, $T(s)$, are in the rhp, in the lhp, and on the $j\omega$-axis:

$$T(s) = \frac{s^3 + 7s^2 - 21s + 10}{s^6 + s^5 - 6s^4 + 0s^3 - s^2 - s + 6}$$

ANSWER:

```
>> routh_hurwitz
Enter highest power in numerator: 3
Enter coefficients, starting with the highest power: 1
Enter coefficients, starting with the highest power: 7
Enter coefficients, starting with the highest power: -21
Enter coefficients, starting with the highest power: 10
Enter highest power in denominator: 6
Enter coefficients, starting with the highest power: 1
Enter coefficients, starting with the highest power: 1
Enter coefficients, starting with the highest power: -6
Enter coefficients, starting with the highest power: 0
Enter coefficients, starting with the highest power: -1
Enter coefficients, starting with the highest power: -1
Enter coefficients, starting with the highest power: 6
Are you entering an open loop or closed loop TF? If closed loop type "C"

G =

      s^3 + 7 s^2 - 21 s + 10
  ------------------------------
  s^6 + s^5 - 6 s^4 - s^2 - s + 6

Continuous-time transfer function.
Model Properties

RH_Table =

     1    -6    -1     6
     1     0    -1     0
    -6     0     6     0
     0     0     0     0
   NaN   NaN   NaN     0
   NaN   NaN   NaN     0
   NaN   NaN   NaN     0


 *You have 1 right-half poles and 5 left-half poles. Thus, you have an u

RH_Table =

     1    -6    -1     6
     1     0    -1     0
    -6     0     6     0
   -18     6     0     0
    -2     6     0     0
   -48     0     0     0
     6     0     0     0


 *Upon changes made to the RH Table, you now have 2 right-half poles and
```

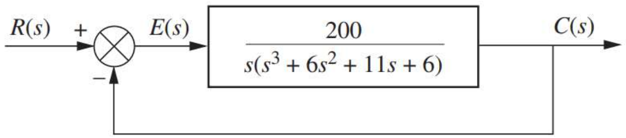
https://jp.mathworks.com/matlabcentral/fileexchange/72884-routh-hurwitz-stability-criteria-table-generator-all-cases

More Examples

上海科技大学 ShanghaiTech University

Standard Routh-Hurwitz

PROBLEM: Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.6.



6

SOLUTION: First, find the closed-loop transfer function as

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200} \qquad (6.14)$$

The Routh table for the denominator of Eq. (6.14) is shown as Table 6.10. For clarity, we leave most zero cells blank. At the s^1 row there is a negative coefficient; thus, there are two sign changes. The system is unstable, since it has two right-half-plane poles and two left-half-plane poles. The system cannot have $j\omega$ poles since a row of zeros did not appear in the Routh table.

TABLE 6.10 Routh table for Example 6.6

s^4	1	11	200
s^3	$\cancel{6}$ 1	$\cancel{6}$ 1	
s^2	$\cancel{10}$ 1	$\cancel{200}$ 20	
s^1	-19		
s^0	20		

Routh-Hurwitz with Zero in First Column

PROBLEM: Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.7.

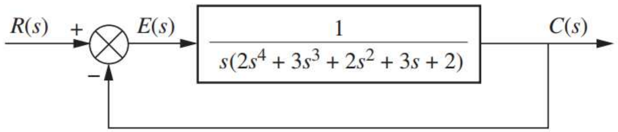


FIGURE 6.7 Feedback control system for Example 6.7

SOLUTION: The closed-loop transfer function is

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1} \qquad (6.15)$$

Form the Routh table shown as Table 6.11, using the denominator of Eq. (6.15). A zero appears in the first column of the s^3 row. Since the entire row is not zero, simply replace

TABLE 6.11 Routh table for Example 6.7

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}$ ϵ	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

More Examples

Stability Design via Routh-Hurwitz

PROBLEM: Find the range of gain, K, for the system of Figure 6.10 that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$.

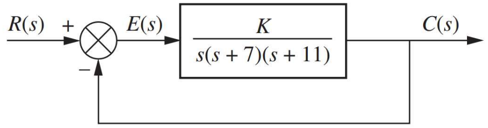


FIGURE 6.10 Feedback control system for Example 6.9

SOLUTION: First find the closed-loop transfer function as

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K} \tag{6.20}$$

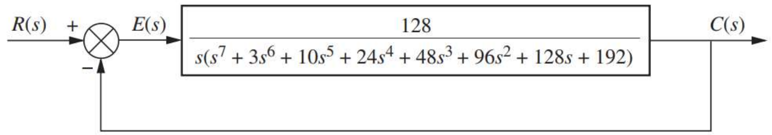
Next form the Routh table shown as Table 6.15.

TABLE 6.15 Routh table for Example 6.9

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Routh-Hurwitz with Row of Zeros

PROBLEM: Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system of Figure 6.8. Draw conclusions about the stability of the closed-loop system.



SOLUTION: The closed-loop transfer function for the system of Figure 6.8 is

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128} \tag{6.17}$$

Using the denominator, form the Routh table shown as Table 6.13. A row of zeros appears in the s^5 row. Thus, the closed-loop transfer function denominator must have an even polynomial as a factor. Return to the s^6 row and form the even polynomial:

$$P(s) = s^6 + 8s^4 + 32s^2 + 64 \tag{6.18}$$

TABLE 6.13 Routh table for Example 6.8

s^8	1	10	48	128	128
s^7	$\cancel{3}$ 1	$\cancel{24}$ 8	$\cancel{96}$ 32	$\cancel{192}$ 64	
s^6	$\cancel{2}$ 1	$\cancel{16}$ 8	$\cancel{64}$ 32	$\cancel{128}$ 64	
s^5	$\cancel{0}$ $\cancel{6}$ 3	$\cancel{0}$ $\cancel{32}$ 16	$\cancel{0}$ $\cancel{64}$ 32	$\cancel{0}$ $\cancel{0}$ 0	
s^4	$\cancel{\frac{8}{3}}$ 1	$\cancel{\frac{64}{3}}$ 8	$\cancel{64}$ 24		
s^3	$\cancel{-8}$ -1	$\cancel{-40}$ -5			
s^2	$\cancel{3}$ 1	$\cancel{24}$ 8			
s^1	3				
s^0	8				

$$\boxed{\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s + 0}$$

More Examples

Factoring via Routh-Hurwitz

PROBLEM: Factor the polynomial

$$s^4 + 3s^3 + 30s^2 + 30s + 200 \tag{6.23}$$

SOLUTION: Form the Routh table of Table 6.17. We find that the s^1 row is a row of zeros. Now form the even polynomial at the s^2 row:

$$P(s) = s^2 + 10 \tag{6.24}$$

TABLE 6.17 Routh table for Example 6.10

s^4	1	30	200
s^3	$\cancel{3}$ 1	$\cancel{30}$ 10	
s^2	$\cancel{20}$ 1	$\cancel{200}$ 10	
s^1	$\cancel{0}$ 2	$\cancel{0}$ 0	
s^0	10		

This polynomial is differentiated with respect to s in order to complete the Routh table. However, since this polynomial is a factor of the original polynomial in Eq. (6.23), dividing Eq. (6.23) by (6.24) yields $(s^2 + 3s + 20)$ as the other factor. Hence,

$$\begin{aligned} s^4 + 3s^3 + 30s^2 + 30s + 200 &= (s^2 + 10)(s^2 + 3s + 20) \\ &= (s + j3.1623)(s - j3.1623) \\ &\quad \times (s + 1.5 + j4.213)(s + 1.5 - j4.213) \end{aligned} \tag{6.25}$$

Stability in State Space

PROBLEM: Given the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u \tag{6.33a}$$

$$y = [1 \quad 0 \quad 0]\mathbf{x} \tag{6.33b}$$

find out how many poles are in the left half-plane, in the right half-plane, and on the $j\omega$-axis.

SOLUTION: First form $(s\mathbf{I} - \mathbf{A})$:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} s & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix} \tag{6.34}$$

Now find the $\det(s\mathbf{I} - \mathbf{A})$:

$$\det(s\mathbf{I} - \mathbf{A}) = s^3 - 6s^2 - 7s - 52 \tag{6.35}$$

Using this polynomial, form the Routh table of Table 6.18.

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix} = \begin{bmatrix} -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{bmatrix} \tag{6.34}$$

Now find the $\det(s\mathbf{I} - \mathbf{A})$:

$$\det(s\mathbf{I} - \mathbf{A}) = s^3 - 6s^2 - 7s - 52 \tag{6.35}$$

Using this polynomial, form the Routh table of Table 6.18.

TABLE 6.18 Routh table for Example 6.11

s^3	1	−7
s^2	$\cancel{-6}$ −3	$\cancel{-52}$ −26
s^1	$\cancel{\frac{47}{3}}$ −1	$\cancel{0}$ 0
s^0	−26	

Steady State Error
(for stable closed loop system)

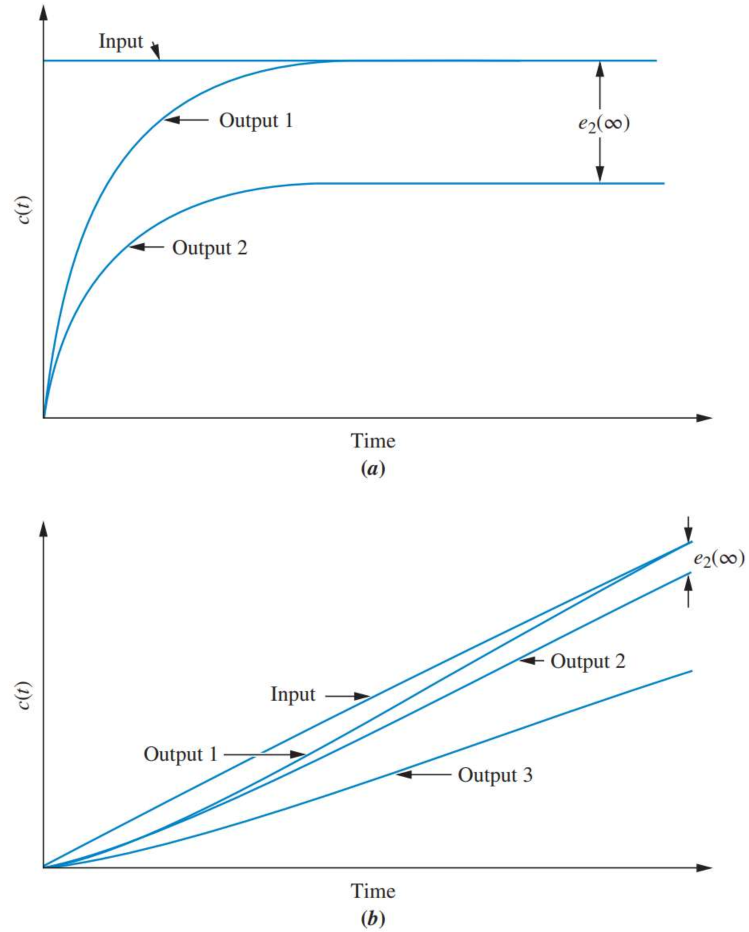


FIGURE 7.2 Steady-state error: **a**. step input; **b**. ramp input

Steady state error

PROBLEM: Find the steady-state error for the system of Figure 7.3(a) if $T(s) = 5/(s^2 + 7s + 10)$ and the input is a unit step.

SOLUTION: From the problem statement, $R(s) = 1/s$ and $T(s) = 5/(s^2 + 7s + 10)$. Substituting into Eq. (7.4) yields

$$E(s) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)} \tag{7.7}$$

$$E(s) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)} \tag{7.7}$$

Since $T(s)$ is stable and, subsequently, $E(s)$ does not have right–half-plane poles or $j\omega$ poles other than at the origin, we can apply the final value theorem. Substituting Eq. (7.7) into Eq. (7.5) gives $e(\infty) = 1/2$.

- **Final value theorem**

$$E(s) = R(s)[1 - T(s)]$$

$$e(\infty) = \lim_{s\to 0} sR(s)[1 - T(s)]$$

[1] The final value theorem is derived from the Laplace transform of the derivative. Thus,

$$\mathscr{L}[\dot{f}(t)] = \int_{0-}^{\infty} \dot{f}(t)e^{-st}dt = sF(s) - f(0-)$$

As $s \to 0$,

$$\int_{0-}^{\infty} \dot{f}(t)dt = f(\infty) - f(0-) = \lim_{s\to 0} sF(s) - f(0-)$$

or

$$f(\infty) = \lim_{s\to 0} sF(s)$$

For finite steady-state errors, the final value theorem is valid only if $F(s)$ has poles only in the left half-plane and, at most, one pole at the origin. However, correct results that yield steady-state errors that are infinite can be obtained if

Steady state error in terms of G(s)

上海科技大学 ShanghaiTech University

- **The steady state performance is evaluated in terms of forward path transfer function**

Step Input. Using Eq. (7.11) with $R(s) = 1/s$, we find

$$\boxed{e(\infty) = e_{\text{step}}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)}}$$

Ramp Input. Using Eq. (7.11), with $R(s) = 1/s^2$, we obtain

$$\boxed{e(\infty) = e_{\text{ramp}}(\infty) = \lim_{s\to 0} \frac{s\left(1/s^2\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s+sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}}$$

Parabolic Input. Using Eq. (7.11), with $R(s) = 1/s^3$, we obtain

$$\boxed{e(\infty) = e_{\text{parabola}}(\infty) = \lim_{s\to 0} \frac{s\left(1/s^3\right)}{1+G(s)} = \lim_{s\to 0} \frac{1}{s^2+s^2G(s)} = \frac{1}{\lim_{s\to 0} s^2G(s)}}$$

$$G(s) \equiv \frac{K(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$$

If there are no integrations, then $n = 0$.

$$\lim_{s\to 0} G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots} K$$

Steady State Error Specifications

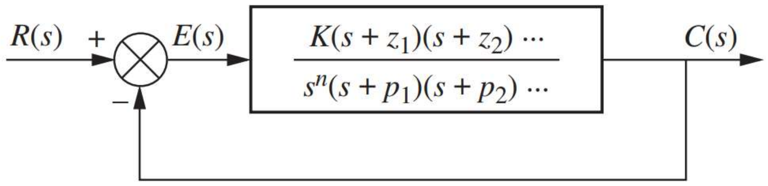


FIGURE 7.8 Feedback control system for defining system type

- **System type**: the number of integrators in the forward path
  - Three error constants are defined for each system type.

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0 $\frac{K(s+z_1)(s+z_2)\cdots}{s^0(s+p_1)(s+p_2)\cdots}$ Static error constant	Type 0 Error	Type 1 $\frac{K(s+z_1)(s+z_2)\cdots}{s^1(s+p_1)(s+p_2)\cdots}$ Static error constant	Type 1 Error	Type 2 $\frac{K(s+z_1)(s+z_2)\cdots}{s^2(s+p_1)(s+p_2)\cdots}$ Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{\lim_{s\to 0} s^2G(s)} = \frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Interpreting the Steady-State Error Specification

PROBLEM: What information is contained in the specification $K_p = 1000$?

上海科技大学 ShanghaiTech University

Gain Design to Meet a Steady-State Error Specification

HC1

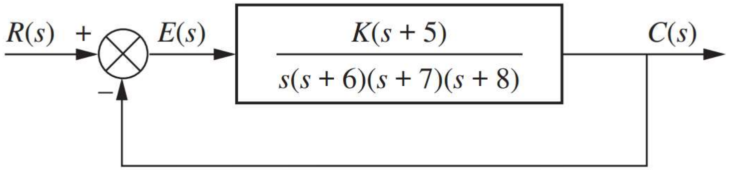


FIGURE 7.10 Feedback control system for Example 7.6

PROBLEM: Given the control system in Figure 7.10, find the value of K so that there is 10% error in the steady state.

SOLUTION: Since the system is Type 1, the error stated in the problem must apply to a ramp input; only a ramp yields a finite error in a Type 1 system. Thus,

$$e(\infty) = \frac{1}{K_v} = 0.1 \tag{7.55}$$

Therefore,

$$K_v = 10 = \lim_{s\to 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8} \tag{7.56}$$

which yields

$$K = 672 \tag{7.57}$$

Applying the Routh-Hurwitz criterion, we see that the system is stable at this gain.

Although this gain meets the criteria for steady-state error and stability, it may not yield a desirable transient response. In Chapter 9 we will design feedback control systems to meet all three specifications.

幻灯片 32

HC1 Hory Chen, 2025/9/30

Steady State Error Due to Disturbance

- The transfer function from input to error is:

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

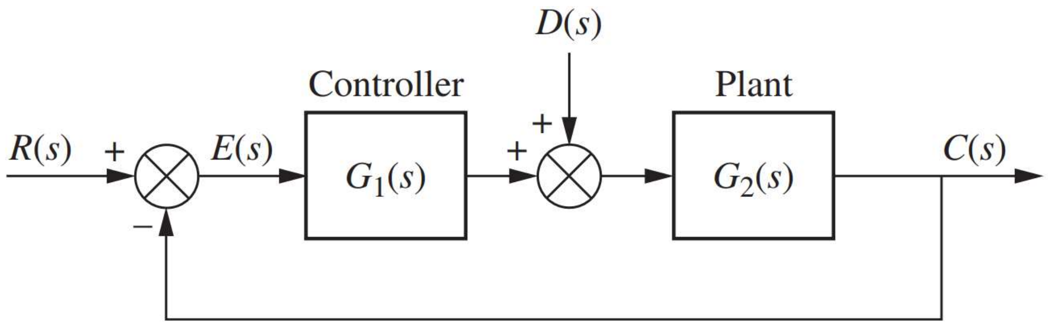


FIGURE 7.11 Feedback control system showing disturbance

$$\begin{aligned} e(\infty) = \lim_{s\to 0} sE(s) &= \lim_{s\to 0} \frac{s}{1 + G_1(s)G_2(s)}R(s) - \lim_{s\to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)}D(s) \\ &= e_R(\infty) + e_D(\infty) \end{aligned}$$

Steady State Error Due to Disturbance

PROBLEM: Find the steady-state error component due to a step disturbance for the system of Figure 7.13.

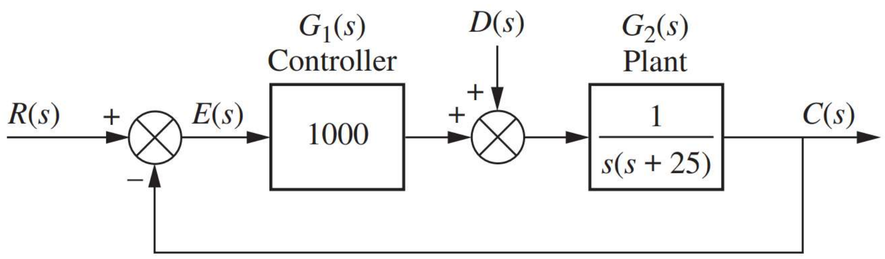


FIGURE 7.13 Feedback control system for Example 7.7

Stability
(in terms of **Root Locus**
for higher order **control systems**)

Control System Problem

- The zeros of T(s) consist of the zeros of G(s) and the poles of H(s).
- The poles of T(s) are not immediately known without factoring the denominator, and they are a function of K.
  - Since the system’s transient response and stability are dependent upon the poles of T(s), we have no knowledge of the system’s performance unless we factor the denominator for specific values of K.
  - We are in need of a analysis tool that gives us a vivid picture of the poles of T(s) as K varies.

$$\boxed{T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}}$$

- E.g., a system with simple open loop poles and zero:

$$G(s)H(s) = \frac{(s+5)}{(s+1)(s+2)(s+3)(s+4)}$$

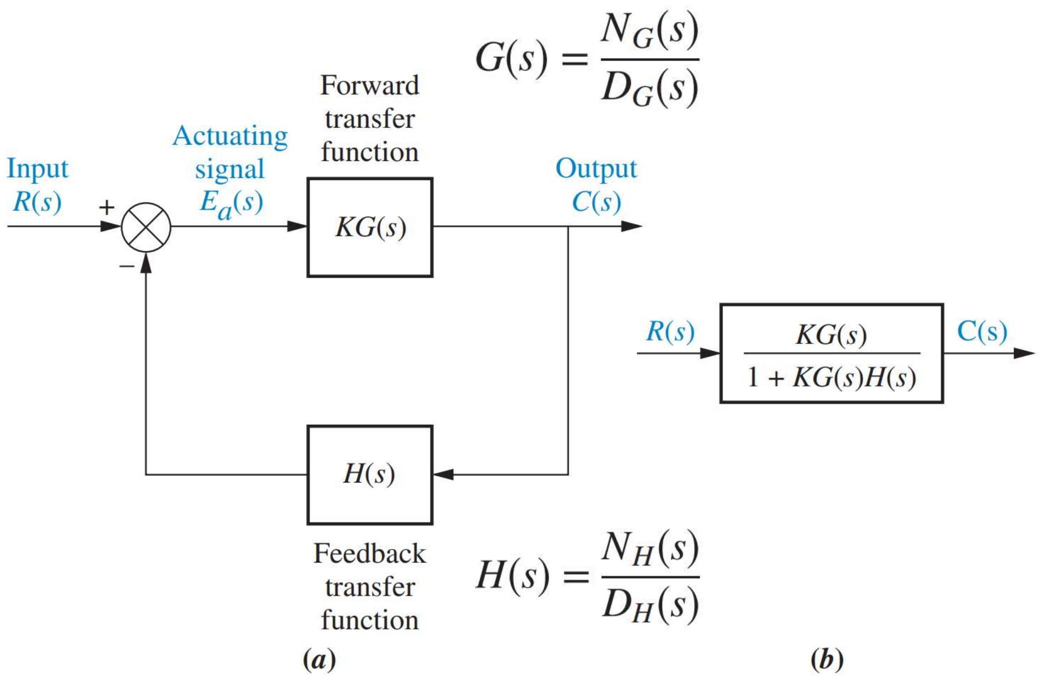


FIGURE 8.1 **a.** Closed-loop system; **b.** equivalent transfer function

Vector Representation of Complex Numbers

- $F(s) = s + a$ is a complex number and can be represented by a vector drawn from **the zero of the function** to **the point s in the complex plane**.
  - It is a very special vector that can be treated as a number.

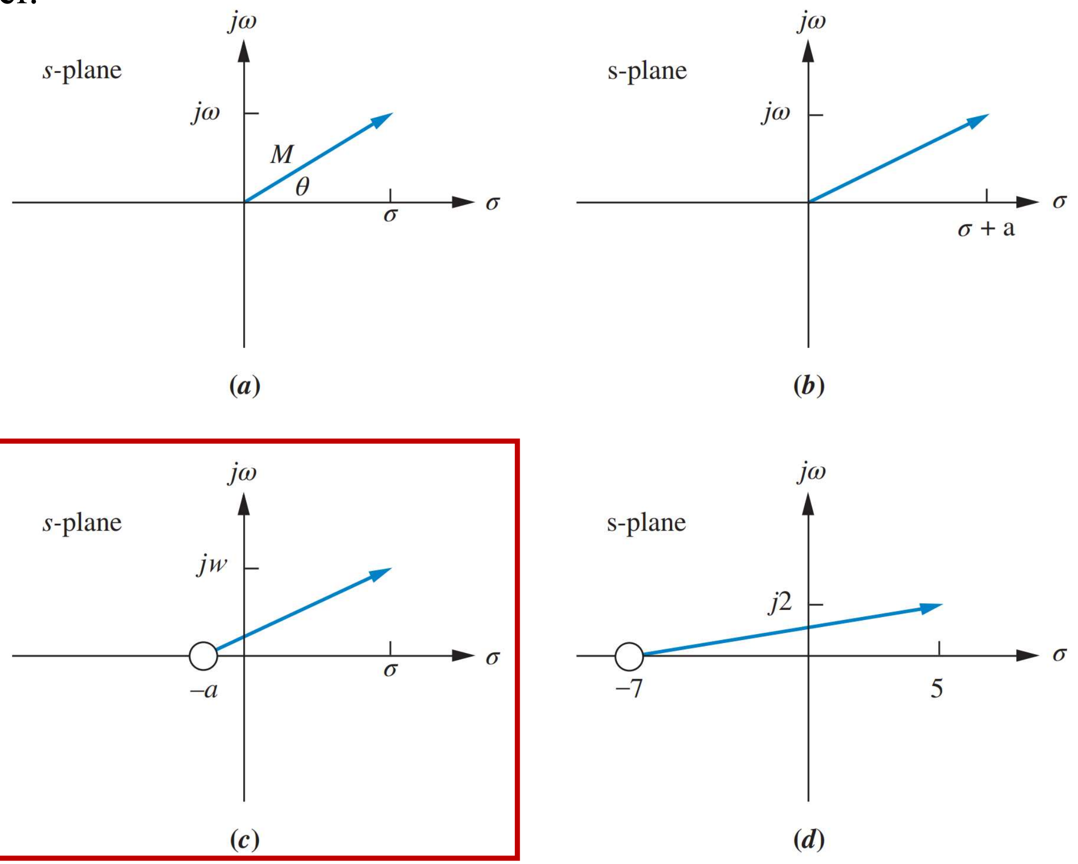


FIGURE 8.2 Vector representation of complex numbers: **a.** $s = \sigma + j\omega$; **b.** $(s + a)$; **c.** alternate representation of $(s + a)$; **d.** $(s + 7)|_{s \to 5+j2}$

上海科技大学 ShanghaiTech University

Evaluation of a Complex Function via Vectors

PROBLEM: Given

$$F(s) = \frac{(s+1)}{s(s+2)} \tag{8.7}$$

find $F(s)$ at the point $s = -3 + j4$.

SOLUTION: The problem is graphically depicted in Figure 8.3, where each vector, $(s+\alpha)$, of the function is shown terminating on the selected point $s = -3 + j4$. The vector originating at the zero at -1 is

$$\sqrt{20}\angle 116.6^\circ \tag{8.8}$$

The vector originating at the pole at the origin is

$$5\angle 126.9^\circ \tag{8.9}$$

The vector originating at the pole at -2 is

$$\sqrt{17}\angle 104.0^\circ \tag{8.10}$$

Substituting Eqs. (8.8) through (8.10) into Eqs. (8.5) and (8.6) yields

$$M\angle\theta = \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^\circ - 126.9^\circ - 104.0^\circ = 0.217\angle - 114.3^\circ \tag{8.11}$$

as the result for evaluating $F(s)$ at the point $-3 + j4$.

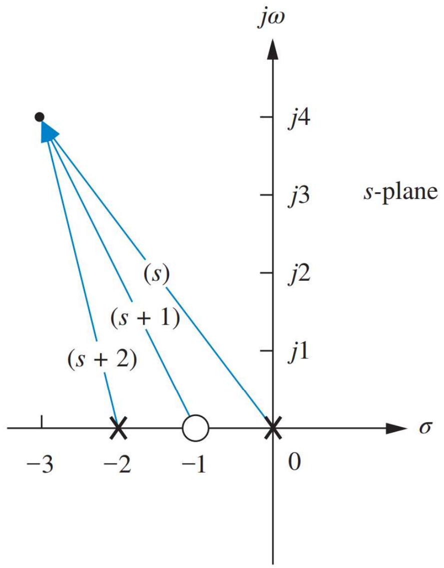


FIGURE 8.3 Vector representation of Eq. (8.7)

$$M = \frac{\Pi \text{ zero lengths}}{\Pi \text{ pole lengths}} = \frac{\prod_{i=1}^{m}|(s+z_i)|}{\prod_{j=1}^{n}|(s+p_j)|}$$

$$\theta = \sum \text{zero angles} - \sum \text{pole angles}$$
$$= \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j)$$

$$F(s) \triangleq G(s)H(s)$$

$$1 + KG(s)H(s) = 0$$

$$K = \frac{1}{|G(s)||H(s)|}$$

Root Locus

上海科技大学 ShanghaiTech University

- Root locus focuses on a simple feedback control system with a proportional gain K.
  - There is variant of root locus method that can deal with problem with more than one parameter.
  - The root locus can be used to analyze closed loop system in terms of information about the open loop transfer function $KG(s)H(s)$

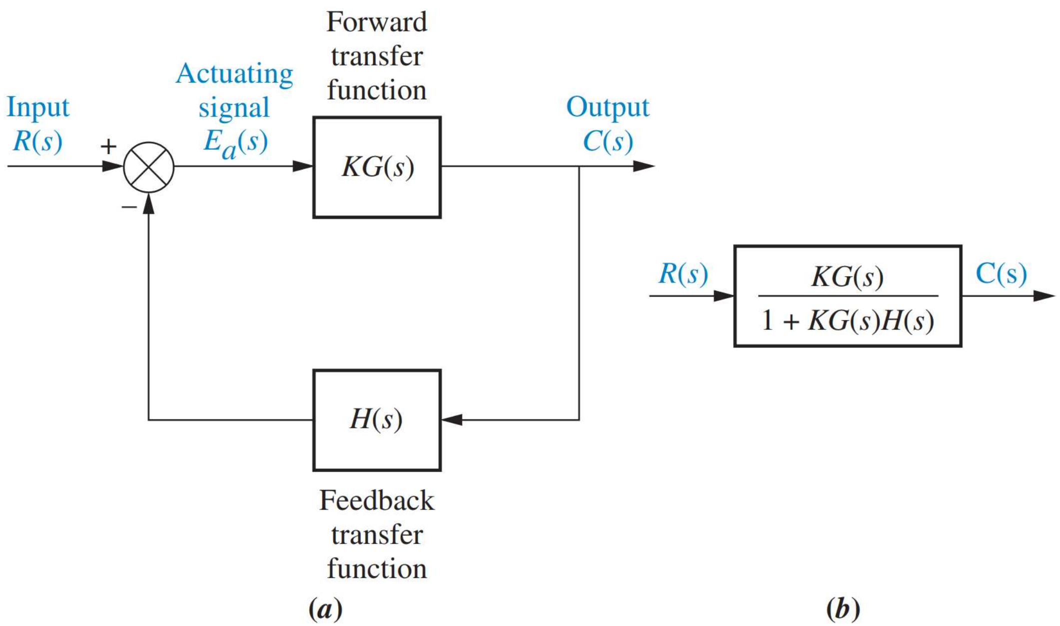


FIGURE 8.1 **a.** Closed-loop system; **b.** equivalent transfer function

Root Locus

- Example root locus with two poles
  - For an undamped time response, the settling time is invariant as gain K varies.

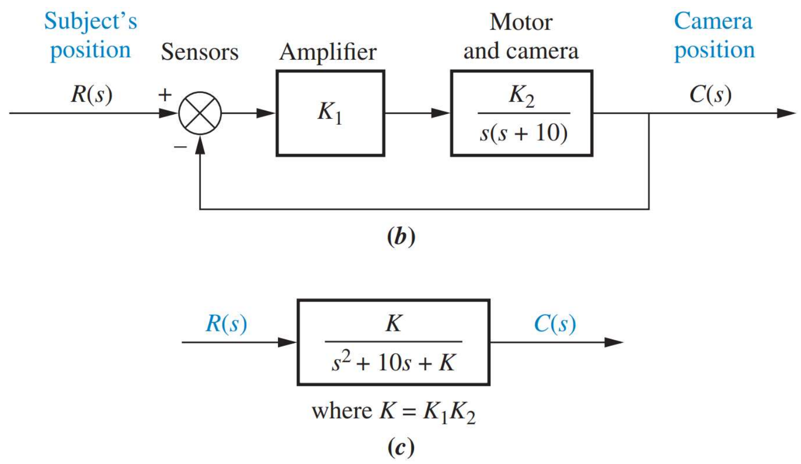


FIGURE 8.4 **a.** Security cameras with auto tracking can be used to follow moving objects automatically; **b.** block diagram; **c.** closed-loop transfer function

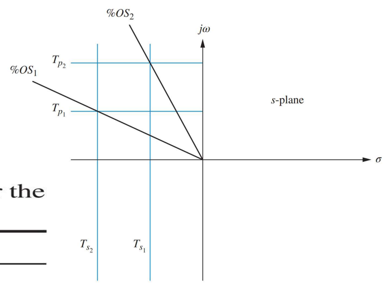


TABLE 8.1 Pole location as function of gain for the system of Figure 8.4

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
20	−7.24	−2.76
25	−5	−5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
40	$-5+j3.87$	$-5-j3.87$
45	$-5+j4.47$	$-5-j4.47$
50	$-5+j5$	$-5-j5$

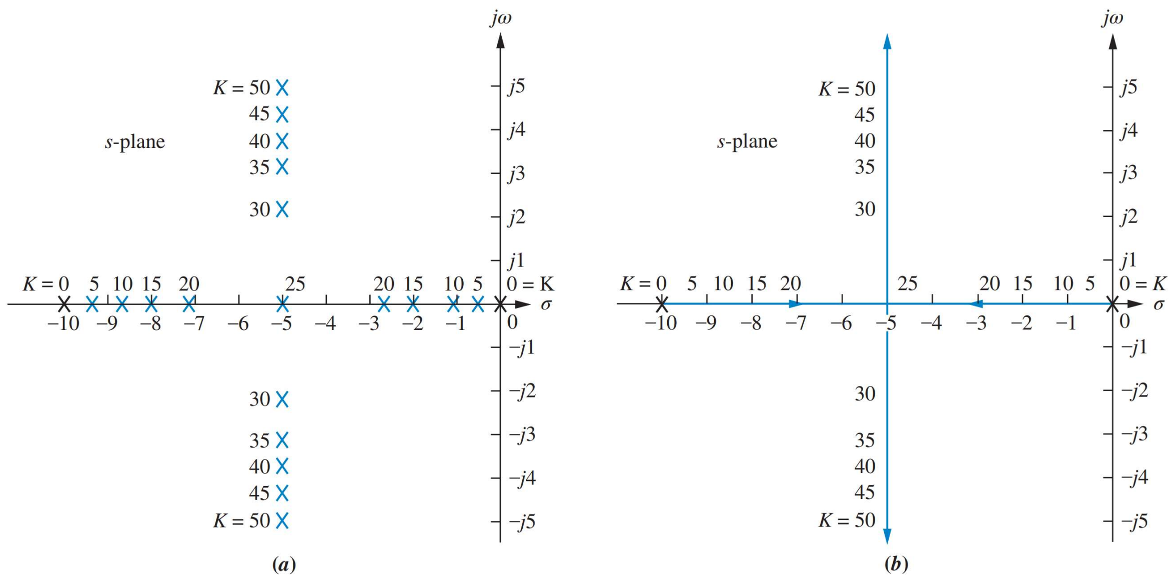


FIGURE 8.5 **a.** Pole plot from Table 8.1; **b.** root locus

Root Locus

- Consider what would happen if that polynomial were of fifth or tenth order. Without a computer, factoring the polynomial would be quite a problem for numerous values of gain.
- We are about to examine the properties of the root locus. From these properties we will be able to make a rapid sketch of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.
  - Motivation: for arbitrary point in complex plane, s, how do we determine it is a pole of the closed loop control system?

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

Closed loop poles

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\angle KG(s)H(s) = (2k+1)180^\circ$$

$$K = \frac{1}{|G(s)||H(s)|}$$

$$1 + K\frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0.$$

$$\prod_{j=1}^{n}(s+p_j) + K\prod_{i=1}^{M}(s+z_i) = 0.$$

- When $K = 0$, the roots of the characteristic equation are the poles of GH(s).
- When $K \to \infty$, the roots of the characteristic equation are the zeros of GH(s).

Sketching Rules

1. The **number of branches** of the root locus equals the number of closed-loop poles.
2. The root locus is **symmetrical** about the real axis.
3. **On the real axis**, the **root locus segment** exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros, (valid for K > 0).
4. The root locus **begins** at the finite and infinite poles of G(s)H(s) and **ends** at the finite and infinite zeros of G(s)H(s).
   1. A function can also have *infinite* poles and zeros.
   2. If the function approaches **infinity** as s approaches infinity, then the function has a pole at infinity. For example, the function $G(s) = s$ has a pole at infinity, since G(s) approaches infinity as s approaches infinity.
   3. If the function approaches **zero** as s approaches infinity, then the function has a zero at infinity. For example, $G(s) = 1/s$ has a zero at infinity, since $G(s)$ approaches zero as s approaches infinity.
5. The root locus approaches **straight lines as asymptotes** as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a. The asymptotes tell us how we get to **zeros/poles at infinity**.

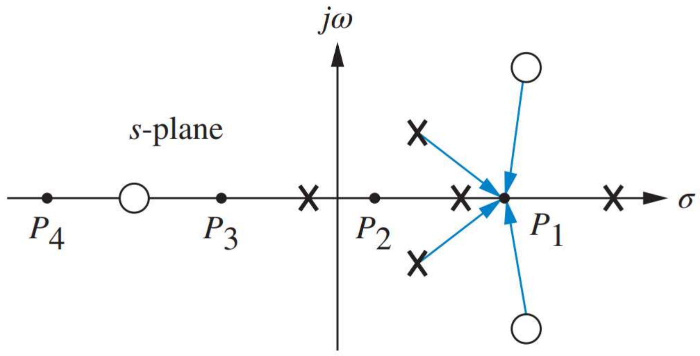


FIGURE 8.8 Poles and zeros of a general open-loop system with test points, P_i, on the real axis

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}$$

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\ \text{finite poles} - \#\ \text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\ \text{finite poles} - \#\ \text{finite zeros}}$$

where $k = 0, \pm 1, \pm 2, \pm 3$

Sketching Rules

PROBLEM: Sketch the root locus for the system shown in Figure 8.11.

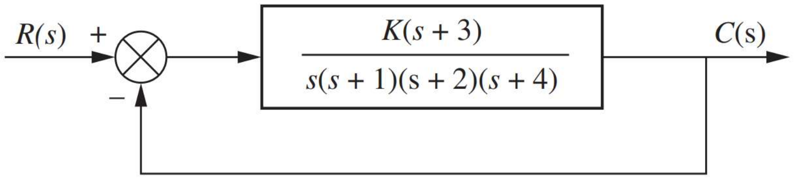


FIGURE 8.11 System for Example 8.2

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

$$= \pi/3 \qquad \text{for } k = 0$$

$$= \pi \qquad \text{for } k = 1$$

$$= 5\pi/3 \qquad \text{for } k = 2$$

$$\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}$$

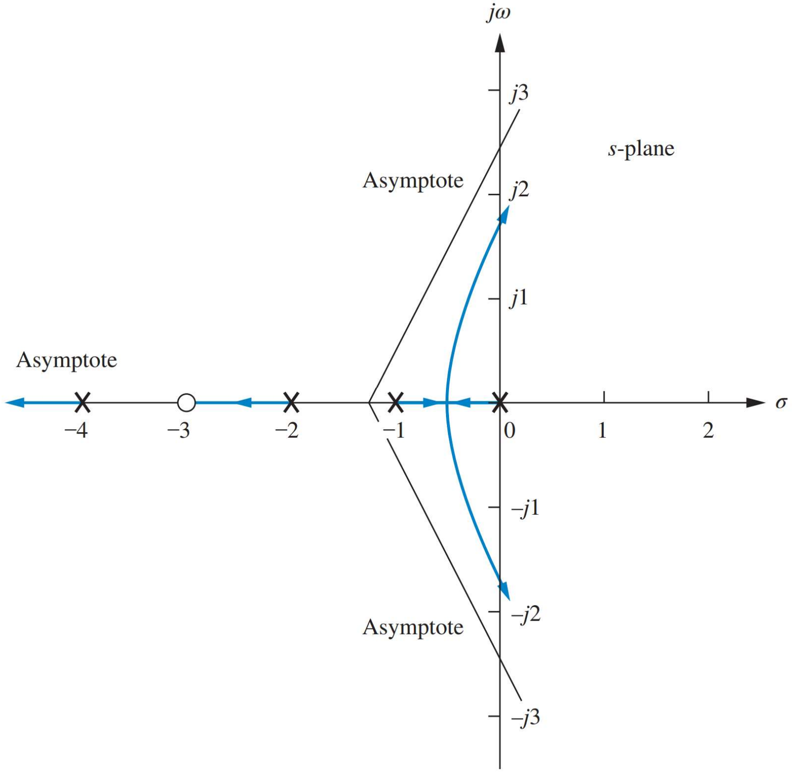


FIGURE 8.12 Root locus and asymptotes for the system of Figure 8.11

Sketching Rules (Refine 1)

- Real-Axis Breakaway and Break-In Points

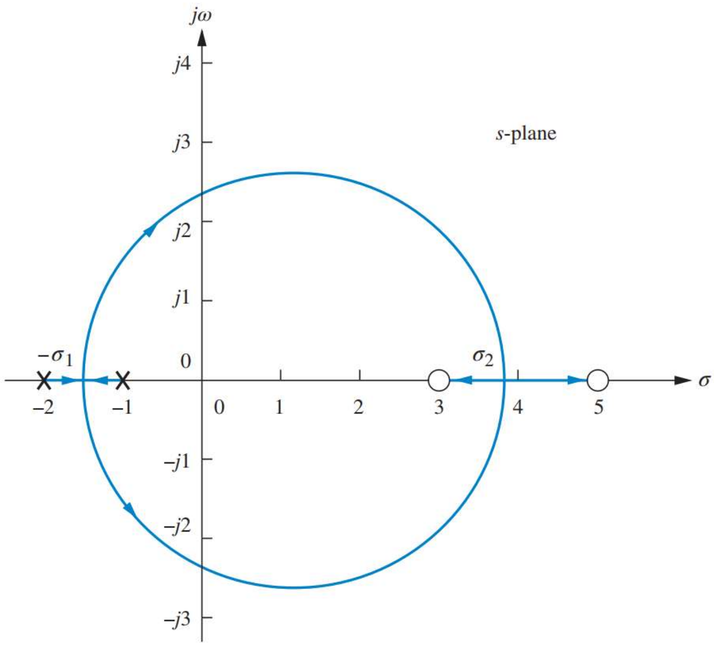


FIGURE 8.13 Root locus example showing real-axis breakaway ($-\sigma_1$) and break-in points (σ_2)

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)} \tag{8.33}$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence,

$$\frac{K(\sigma^2-8\sigma+15)}{(\sigma^2+3\sigma+2)} = -1 \tag{8.34}$$

Solving for K, we find

$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)} \tag{8.35}$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2-26\sigma-61)}{(\sigma^2-8\sigma+15)^2} = 0 \tag{8.36}$$

Solving for σ, we find $\sigma = -1.45$ and 3.82, which are the breakaway and break-in points.

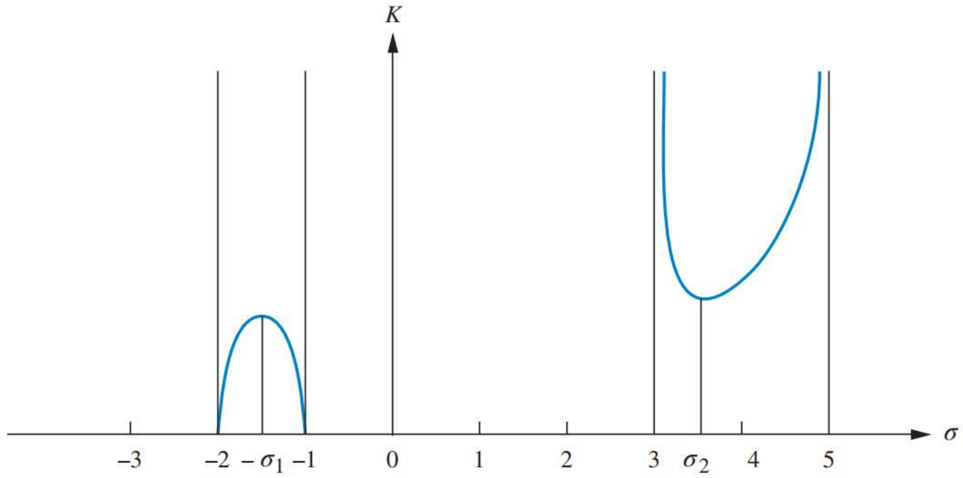


FIGURE 8.14 Variation of gain along the real axis for the root locus of Figure 8.13

Sketching Rules (Refine 2)

- The jω-Axis Crossings
  - To find the $j\omega$-axis crossing, we can use the Routh-Hurwitz criterion, covered in Chapter 6, as follows: Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

TABLE 8.3 Routh table for Eq (8.40)

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\dfrac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

Frequency and Gain at Imaginary-Axis Crossing

PROBLEM: For the system of Figure 8.11, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?

SOLUTION: The closed-loop transfer function for the system of Figure 8.11 is

$$T(s) = \frac{K(s+3)}{s^4+7s^3+14s^2+(8+K)s+3K} \tag{8.40}$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in Table 8.3.

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2-65K+720=0 \tag{8.41}$$

From this equation K is evaluated as

$$K=9.65 \tag{8.42}$$

Forming the even polynomial by using the s^2 row with $K=9.65$, we obtain

$$(90-K)s^2+21K = 80.35s^2+202.7=0 \tag{8.43}$$

and s is found to be equal to $\pm j1.59$. Thus the root locus crosses the $j\omega$-axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \le K < 9.65$.

Sketching Rules (Refine 3)

- Angles of Departure/Arrival

Angle of Departure from a Complex Pole

PROBLEM: Given the unity feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

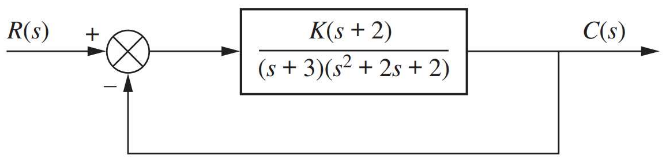


FIGURE 8.16 Unity feedback system with complex poles

SOLUTION: Using the poles and zeros of $G(s) = (s+2)/[(s+3)(s^2+2s+2)]$ as plotted in Figure 8.17, we calculate the sum of angles drawn to a point ϵ close to the complex pole, $-1+j1$, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ \qquad (8.46)$$

from which $\theta = -251.6^\circ = 108.4^\circ$. A sketch of the root locus is shown in Figure 8.17. Notice how the departure angle from the complex poles helps us to refine the shape.

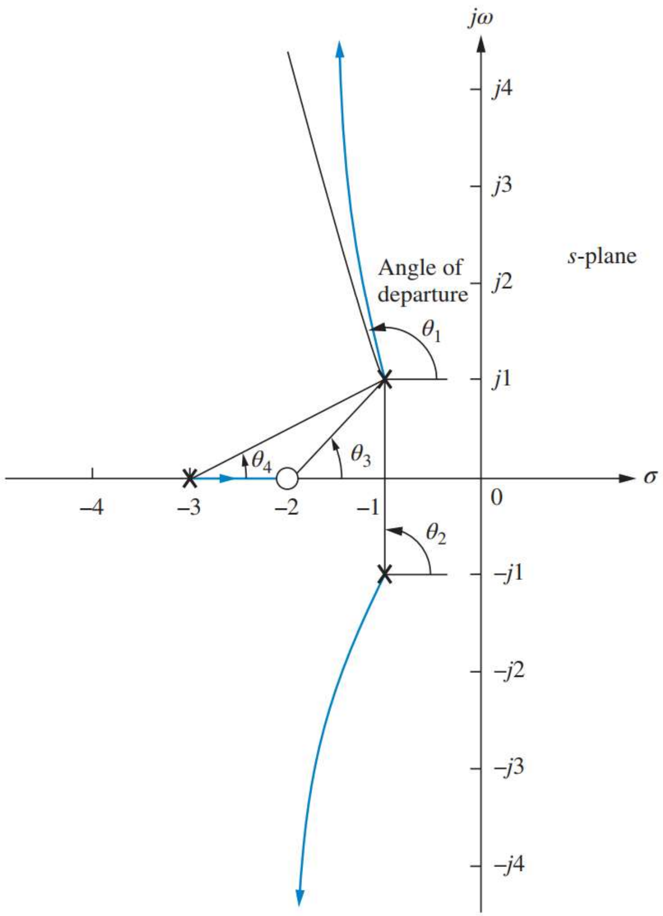


FIGURE 8.17 Root locus for system of Figure 8.16 showing angle of departure

Sketching Rules (Refine 4)

- Sometimes we need to searching root locus on constant damping ratio line, and Routh table is no longer helpful in this case
- To find the crossing point of the root locus and a constant damping ratio line, select a point, e.g., the point at radius $r = 2$ on the $\zeta = 0.45$ line, and add the angles of the zeros and subtract the angles of the poles, obtaining:

$$\theta_2 - \theta_1 - \theta_3 - \theta_4 - \theta_5 = -251.5°$$

which suggests we need to reduce the radius and select another point, until the angle summation gives an angle of 180 degrees, and at that point we calculate for the gain by:

$$K = \frac{|A||C||D||E|}{|B|} = 1.71$$

Radius r	Angle $\angle$ (degrees)
0.5	–158.4
0.747	–180.0
1.0	–199.9
1.5	–230.4
2.0	–251.5

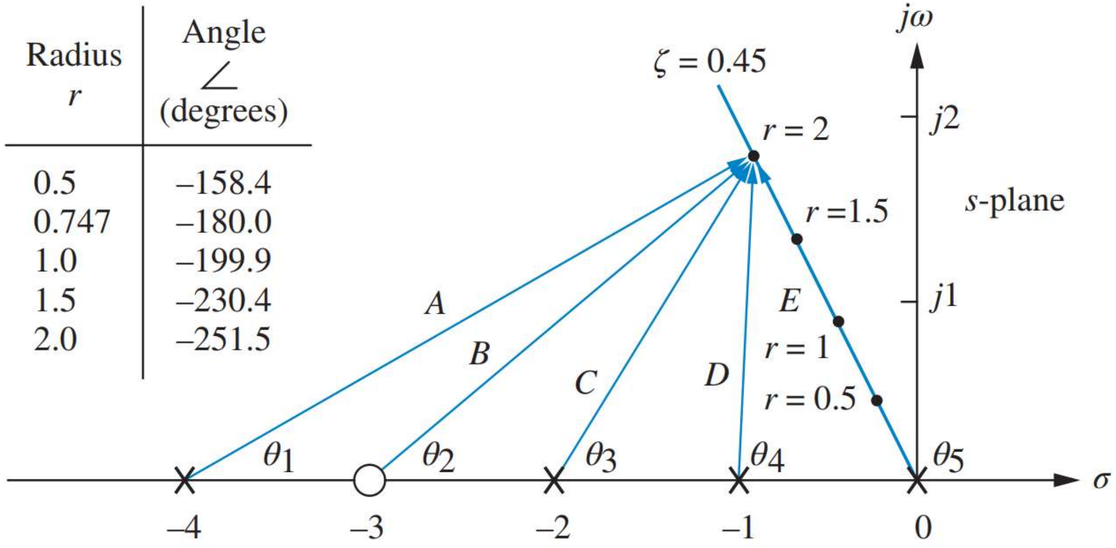


FIGURE 8.18 Finding and calibrating exact points on the root locus of Figure 8.12

Summary

上海科技大学 ShanghaiTech University

Basic Rules for Sketching the Root Locus

Number of branches The number of branches of the root locus equals the number of closed-loop poles.

Symmetry The root locus is symmetrical about the real axis.

Real-axis segments On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Starting and ending points The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Behavior at infinity The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes are given by the real-axis intercept and angle in radians as follows:

$$\boxed{\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}} \tag{8.49}$$

$$\boxed{\theta_a = \frac{(2k+1)\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}} \tag{8.50}$$

where $k = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots$.

Additional Rules for Refining the Sketch

Real-axis breakaway and break-in points The root locus breaks away from the real axis at a point where the gain is maximum and breaks into the real axis at a point where the gain is minimum.

Calculation of $j\omega$-axis crossings The root locus crosses the $j\omega$-axis at the point where $\angle G(s)H(s) = (2k+1)180°$. Routh-Hurwitz or a search of the $j\omega$-axis for $(2k+1)180°$ can be used to find the $j\omega$-axis crossing.

Angles of departure and arrival The root locus departs from complex, open-loop poles and arrives at complex, open-loop zeros at angles that can be calculated as follows. Assume a point ϵ close to the complex pole or zero. Add all angles drawn from all open-loop poles and zeros to this point. The sum equals $(2k+1)180°$. The only unknown angle is that drawn from the ϵ close pole or zero, since the vectors drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is ϵ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

Plotting and calibrating the root locus All points on the root locus satisfy the relationship $\angle G(s)H(s) = (2k+1)180°$. The gain, K, at any point on the root locus is given by

$$\boxed{K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} - \frac{\Pi\ \text{finite pole lengths}}{\Pi\ \text{finite zero lengths}}} \tag{8.51}$$

Root Locus Sketch

Sketching a Root Locus and Finding Critical Points

PROBLEM: Sketch the root locus for the system shown in Figure 8.19(a) and find the following:

a. The exact point and gain where the locus crosses the 0.45 damping ratio line

b. The exact point and gain where the locus crosses the $j\omega$-axis

c. The breakaway point on the real axis

d. The range of K within which the system is stable

$3.4 \angle 116.7°$

K, of 0.417.

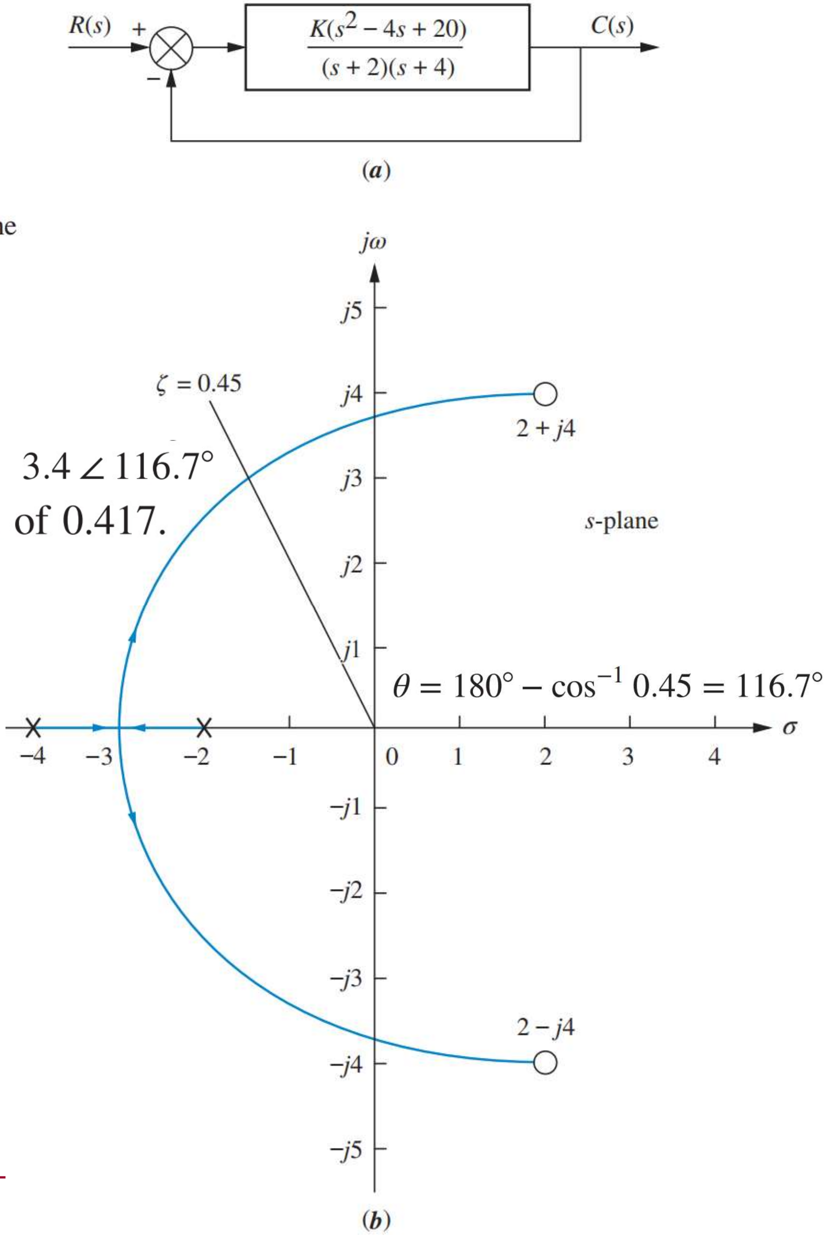


FIGURE 8.19 **a.** System for Example 8.7; **b.** root locus sketch

Root Locus Sketch

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2 - 4s + 13)}$$

do the following:

a. Sketch the root locus.

b. Find the imaginary-axis crossing.

c. Find the gain, K, at the $j\omega$-axis crossing.

d. Find the break-in point.

e. Find the angle of departure from the complex poles.

ANSWERS:

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s-2)(s-4)}{(s^2 + 6s + 25)}$$

do the following:

a. Sketch the root locus.

b. Find the imaginary-axis crossing.

c. Find the gain, K, at the $j\omega$-axis crossing.

d. Find the break-in point.

e. Find the point where the locus crosses the 0.5 damping ratio line.

f. Find the gain at the point where the locus crosses the 0.5 damping ratio line.

g. Find the range of gain, K, for which the system is stable.

ANSWERS:

Generalized Root Locus

上海科技大学 ShanghaiTech University

- When the parameter does not appear as the proportional gain, we can still find a way to draw the root locus, by isolating the parameter

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{10}{s^2 + (p_1 + 2)s + 2p_1 + 10} \qquad (8.59)$$

$$T(s) = \frac{10}{s^2 + 2s + 10 + p_1(s+2)}$$

$$T(s) = \frac{\dfrac{10}{s^2 + 2s + 10}}{1 + \dfrac{p_1(s+2)}{s^2 + 2s + 10}}$$

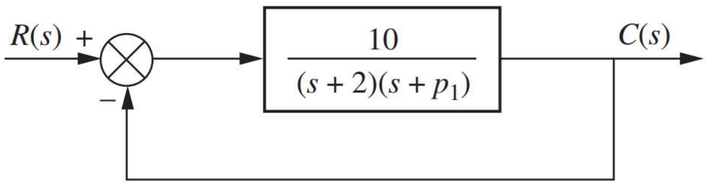


FIGURE 8.24 System requiring a root locus calibrated with p_1 as a parameter

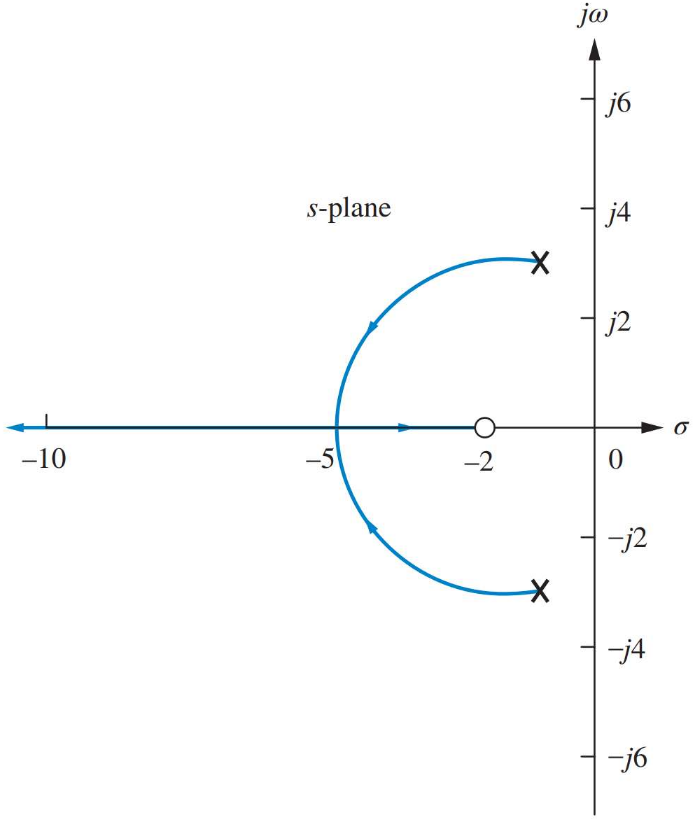


FIGURE 8.25 Root locus for the system of Figure 8.24, with p_1 as a parameter

Generalized Root Locus

PROBLEM: Sketch the root locus for variations in the value of p_1, for a unity feedback system that has the following forward transfer function:

$$G(s) = \frac{100}{s(s + p_1)}$$

Generalized Root Locus (Two Parameters)

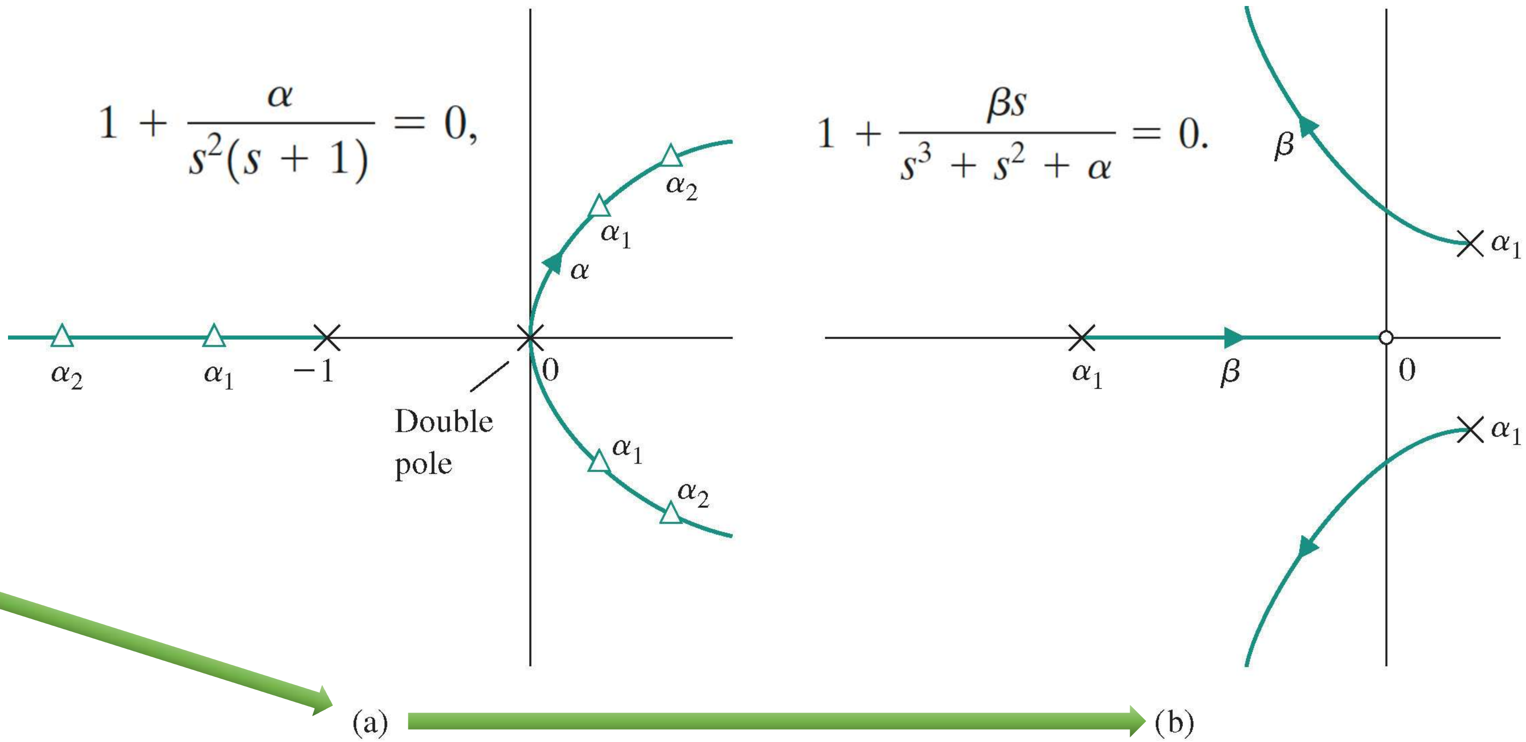
- Consider a characteristic equation with two tuning parameters:

$$s^3 + s^2 + \beta s + \alpha = 0.$$

$$1 + \frac{\beta s}{s^3 + s^2 + \alpha} = 0.$$

$$s^3 + s^2 + \alpha = 0,$$

$$1 + \frac{\alpha}{s^2(s+1)} = 0,$$



Root loci as a function of α and β. (a) Loci as α varies. (b) Loci as β varies for one value of $\alpha = \alpha_1$.

Generalized Root Locus (Two Parameters)

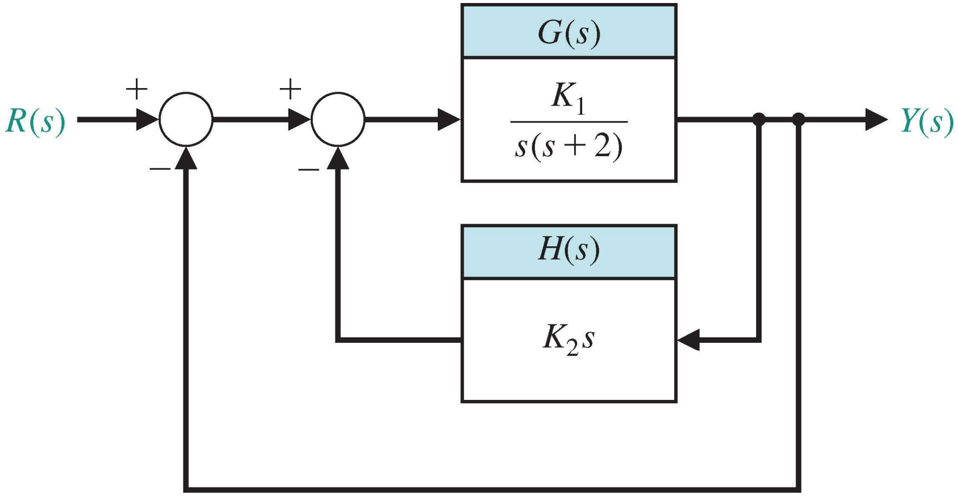
- Example: Welding head control

- The control specifications

  1. Steady-state error for a ramp input is $e_{ss} \le 35\%$ of input slope

  2. Damping ratio of dominant roots is $\zeta \ge 0.707$

  3. Settling time to within 2% of the final value is $T_s \le 3$ s



- The control parameter to be designed

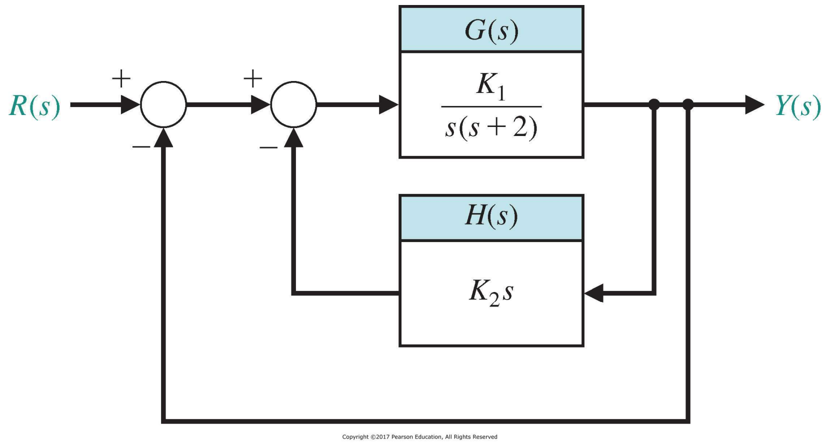
  The amplifier gain K_1 and the derivative feedback gain K_2 are to be selected

Generalized Root Locus (Two Parameters)

- Closed-loop system

$$G_2(s) = G(s)/(1 + G(s)H(s)) = \frac{K_1}{s^2 + (K_1K_2 + 2)s}$$

$$T(s) = \frac{G_2(s)}{1 + G_2(s)} = \frac{K_1}{s^2 + (K_1K_2 + 2)s + K_1}$$



- The control specifications

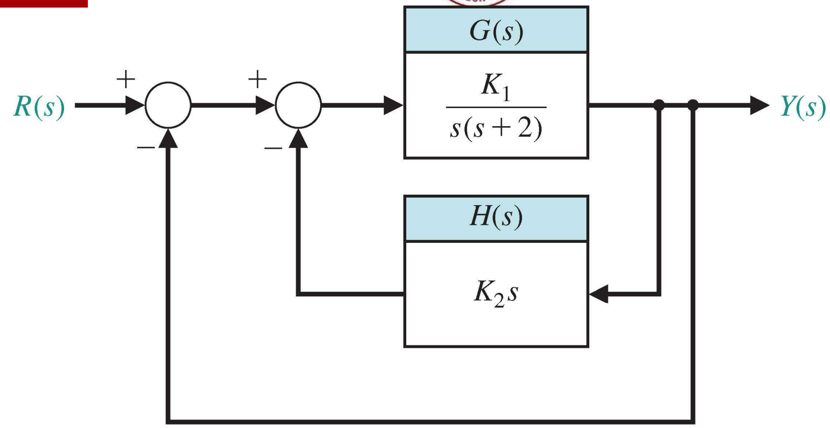
1. Steady-state error for a ramp input is $e_{ss} \le 35\%$ of input slope

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(|R|/s^2)}{1 + G_2(s)},$$

$$\frac{e_{ss}}{|R|} = \frac{2 + K_1K_2}{K_1} \le 0.35.$$

indicates we need to select a small K_2, a large K_1 to achieve a low value of steady-state error.

Generalized Root Locus (Two Parameters)



- Closed-loop system

$$G_2(s) = G(s)/(1 + G(s)H(s)) = \frac{K_1}{s^2 + (K_1K_2 + 2)s}$$

$$T(s) = \frac{G_2(s)}{1 + G_2(s)} = \frac{K_1}{s^2 + (K_1K_2 + 2)s + K_1}$$

- The control specifications

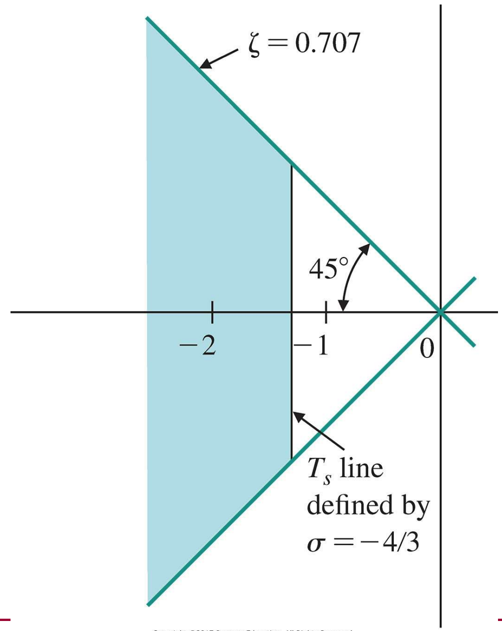
  2. Damping ratio of dominant roots is $\zeta \geq 0.707$

  3. Settling time to within 2% of the final value is $T_s \leq 3$ s

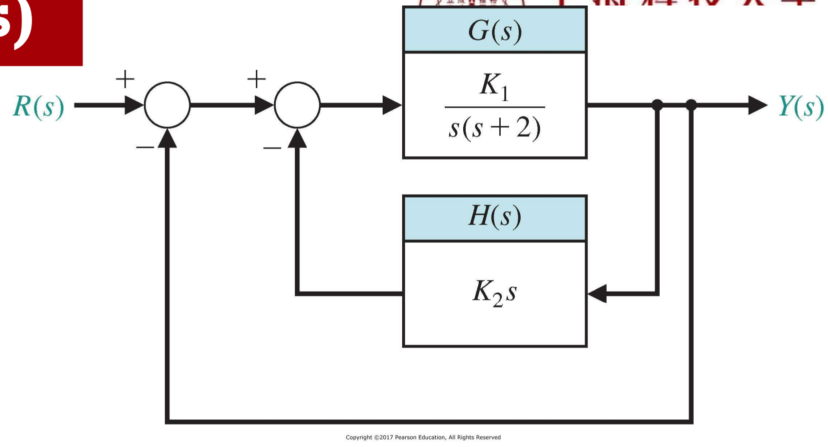
$$\xi = \frac{K_1K_2+2}{2\sqrt{K_1}} \geq 0.707$$

The settling time specification can be rewritten in terms of the real part of the dominant roots as

$$T_s = \frac{4}{\sigma_d} \leq 3 \text{ s.} \quad \Rightarrow \quad \sigma_d \geq 4/3$$



Generalized Root Locus (Two Parameters)



The characteristic equation is

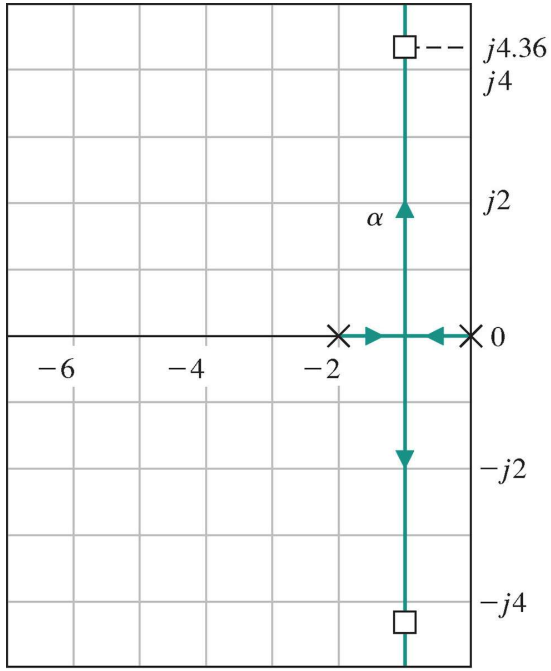
$$s^2 + 2s + \beta s + \alpha = 0.$$

with $\alpha = K_1$ and $\beta = K_2 K_1$.

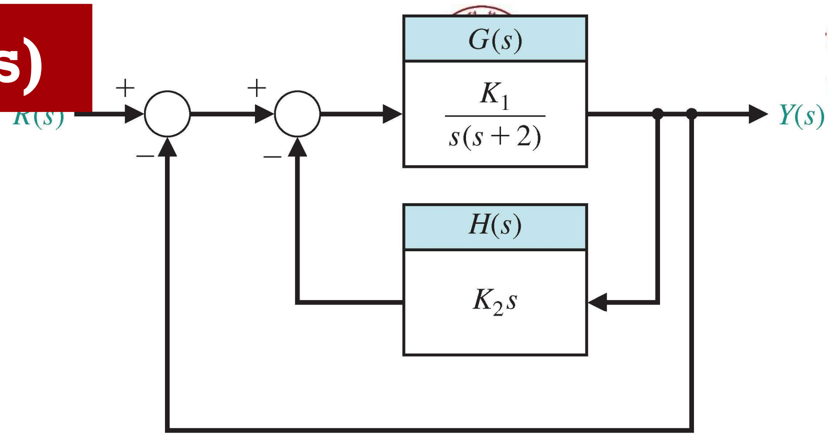
The locus of roots as $\alpha = K_1$ varies (set $\beta = 0$) is determined from the equation

$$1 + \frac{\alpha}{s(s+2)} = 0,$$

For a gain of $K_1 = \alpha = 20$, the roots are $s = -1 \pm j4.36$



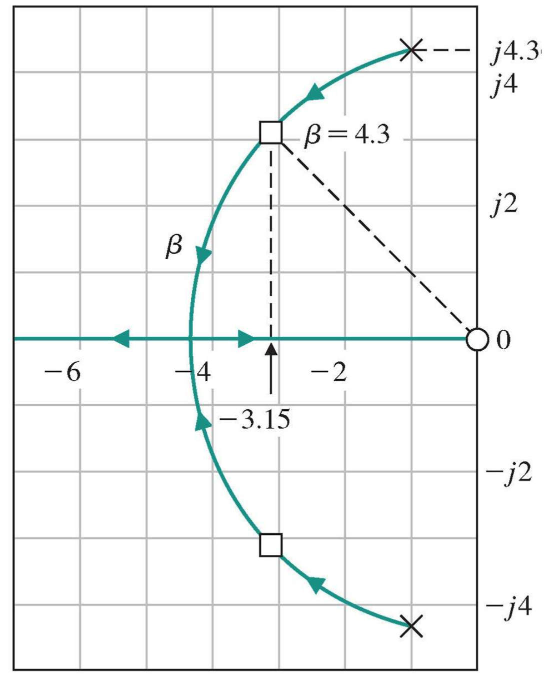
Generalized Root Locus (Two Parameters)



The characteristic equation is

$$s^2 + 2s + \beta s + \alpha = 0.$$

with $\alpha = K_1$ and $\beta = K_2 K_1$.



Then the effect of varying $\beta = 20K_2$ is determined from

$$1 + \frac{\beta s}{s^2 + 2s + 20} = 0.$$

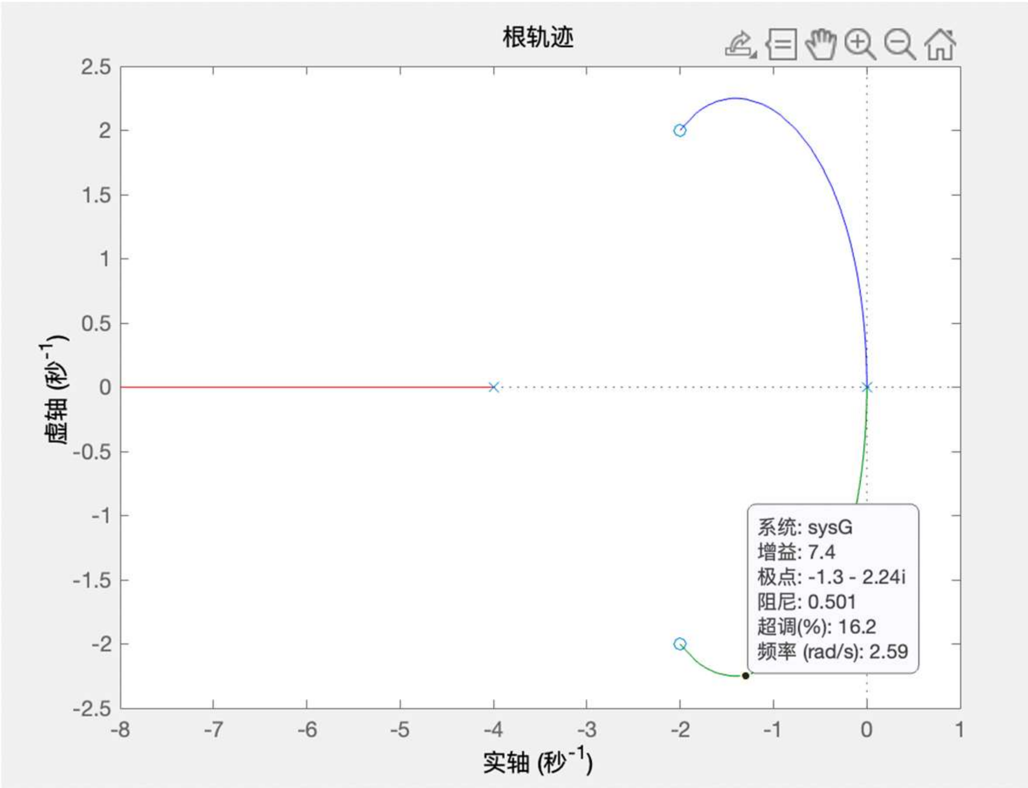
- Roots with $\zeta = 0.707$ are obtained when $\beta = 4.3 = 20K_2$ or when $K_2 = 0.215$.
- The real part of these roots is $\sigma = -3.15$;

Exercise

E1. A unity feedback control system for an automobile suspension tester has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s^2 + 4s + 8)}{s^2(s + 4)}.$$

We desire the dominant roots to have a ζ around 0.5. Using the root locus, to find the proper value of K.



$$K = 7.35$$

the dominant roots are $s = -1.3 \pm j2.2.$

Sensitivity

Based upon the previous discussion, let us formalize a definition of sensitivity: *Sensitivity* is the ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero. That is,

$$
\begin{aligned}
S_{F:P} &= \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function, } F}{\text{Fractional change in the parameter, } P} \\
&= \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} \\
&= \lim_{\Delta P \to 0} \frac{P \Delta F}{F \Delta P}
\end{aligned}
$$

which reduces to

$$\boxed{S_{F:P} = \frac{P}{F}\frac{\delta F}{\delta P}}$$

Sensitivity of a Closed-Loop Transfer Function

PROBLEM: Given the system of Figure 7.19, calculate the sensitivity of the closed-loop transfer function to changes in the parameter a. How would you reduce the sensitivity?

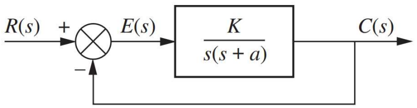


FIGURE 7.19 Feedback control system for Examples 7.10 and 7.11

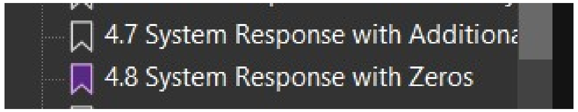
SOLUTION: The closed-loop transfer function is

$$T(s) = \frac{K}{s^2 + as + K} \tag{7.76}$$

Using Eq. (7.75), the sensitivity is given by

$$S_{T:a} = \frac{a}{T}\frac{\delta T}{\delta a} = \frac{a}{\left(\dfrac{K}{s^2 + as + K}\right)}\left(\frac{-Ks}{(s^2 + as + K)^2}\right) = \frac{-as}{s^2 + as + K} \tag{7.77}$$

which is, in part, a function of the value of s. For any value of s, however, an increase in K reduces the sensitivity of the closed-loop transfer function to changes in the parameter a.


4.7 System Response with Additiona
4.8 System Response with Zeros

Dominant Pole Concept

Response of systems with zeros