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Lecture 2: Mathematical Models

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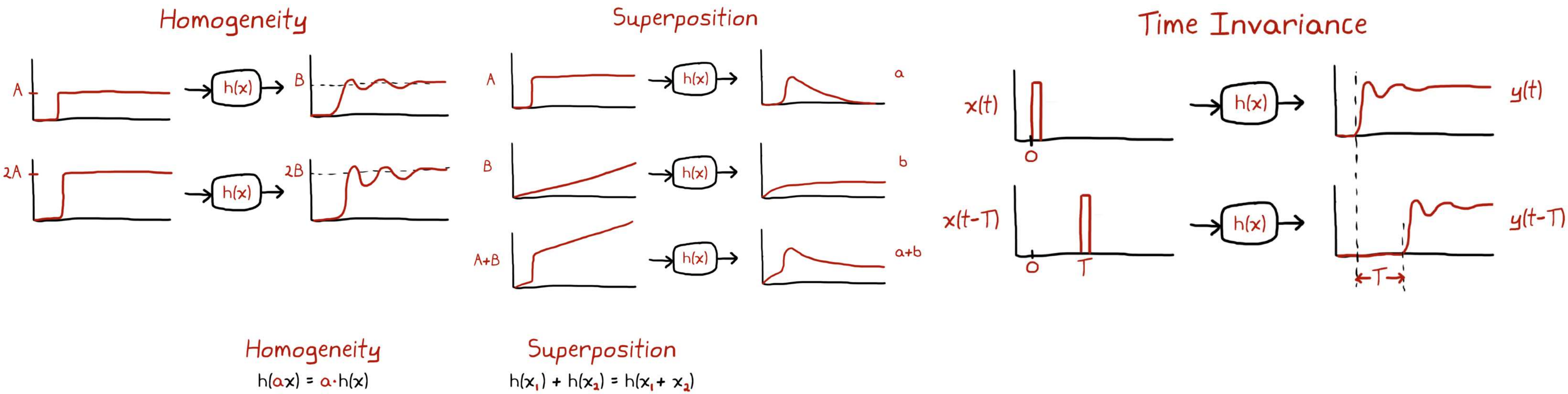
SIST 1D#206

Linear System

Linear System

- Linearity is defined in terms of the input $u(t)$ and output $y(t)$
  - Also known as excitation and response.
- If a system satisfies **homogeneity** and **superposition**, it is a linear system.
- If a linear system further sanctifies the property of **time-invariance**, it is then called an LTI system.
  - Linearity + Time-invariance = **sinusoidal fidelity**
  - Linear system is a wider concept than *linear time-invariant (LTI)* system, also including *linear time-varying* system



Linear Operator

- Linear time-invariant operator

LTI Allowable operations:

Multiply or divide the input by a constant	Integrate or differentiate the input	Add or subtract multiple inputs
$a \cdot x(t)$ $\frac{1}{a} \cdot x(t)$	$\int x(t)dt$ $\frac{d\,x(t)}{dt}$	$x_1(t) + x_2(t)$ $x_1(t) - x_2(t)$

- Generally speaking, a system is also an operator (we are not ready to see this yet)

True or False:

The differential equation $\frac{\mathrm{d}}{\mathrm{d}t}y = -ty + u$ describes a linear system.

Laplace Transform

Excitation Functions

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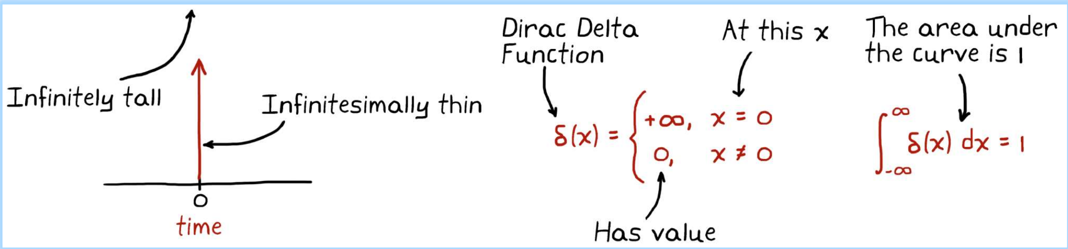
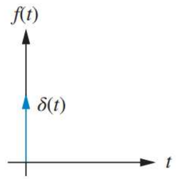
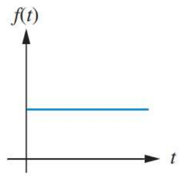
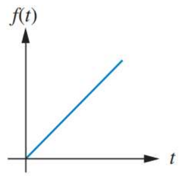
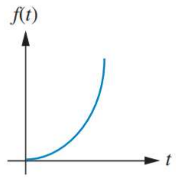
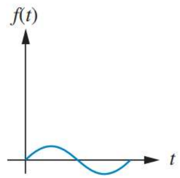
- Dirac delta function $\delta(t)$



- Heaviside step function $u(t)$

$$u(t) = 1 \quad t > 0$$
$$\quad\;\; = 0 \quad t < 0$$

TABLE 1.1 Test waveforms used in control systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

Laplace Transform

- Laplace transform is an integral transform with a fast decaying kernel function,
  - because the most important thing for an integral is to exist

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Kernel is: $K(s,t) = e^{-st}$

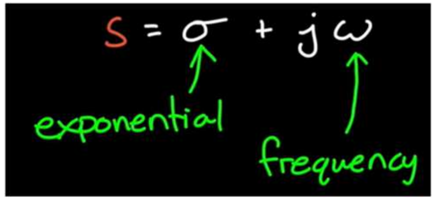


TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$ [$u(t) = 1 \quad t > 0$; $= 0 \quad t < 0$]	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Example 2.1

Laplace Transform of a Time Function

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt = \int_0^{\infty} Ae^{-at}e^{-st}\,dt = A\int_0^{\infty} e^{-(s+a)t}\,dt$$
$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a} \qquad (2.3)$$

- Inverse Laplace transform is often done with further help of the following table

Example 2.2

Inverse Laplace Transform

PROBLEM: Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION: For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

TABLE 2.2 Laplace transform theorems

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)]$	$= F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Norman Nise , Control system engineering (7th Edition)

Laplace Transform

- The two tables can be used to derive for more complicated results, e.g.,

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$

TABLE 2.1 Laplace transform table

Item no.	*f(t)*	*F(s)*
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
4.	$\mathscr{L}[e^{-at} f(t)] = F(s + a)$	Frequency shift theorem

$$\mathscr{L}[Ae^{-at} \cos \omega t] = \frac{A(s + a)}{(s + a)^2 + \omega^2}$$

Laplace Transform

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- Laplace transform converts a differential equation into algebraic equation for its **differentiation theorem**

TABLE 2.2 Laplace transform theorems

Item no.	Theorem		Name
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem

Example 2.3

Laplace Transform Solution of a Differential Equation

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \tag{2.14}$$

Laplace Transform

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PROBLEM: Find the Laplace transform of $f(t) = te^{-5t}$.

ANSWER:

PROBLEM: Find the inverse Laplace transform of $F(s) = 10/[s(s+2)(s+3)^2]$.

ANSWER:

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$ — $u(t) = 1 \quad t > 0$; $= 0 \quad t < 0$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Transfer Function

Transfer Function

- General case of a high order differential equation:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t) \quad (2.50)$$

- **Convert it into s-domain:**

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t)$$
$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t)$$

- Assume zero initial conditions we get transfer function

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)}$$

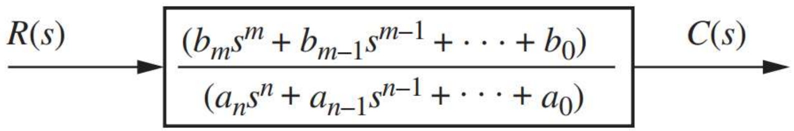


FIGURE 2.2 Block diagram of a transfer function

Norman Nise , Control system engineering (7th Edition)

Transfer Function

PROBLEM: Find the transfer function, $G(s) = C(s)/R(s)$, corresponding to the differential equation $\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$.

ANSWER:

PROBLEM: Find the differential equation corresponding to the transfer function,

$$G(s) = \frac{2s + 1}{s^2 + 6s + 2}$$

ANSWER:

PROBLEM: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s + 4)(s + 8)}$$

ANSWER:

System Examples

Circuit

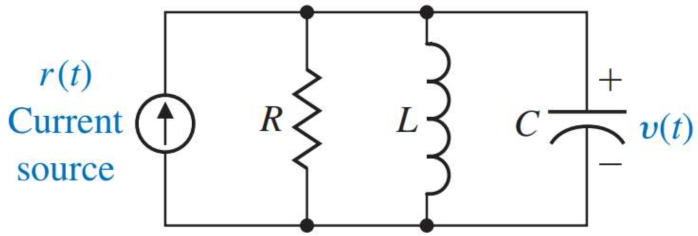


FIGURE 2.3
RLC circuit.

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\, dt = r(t)$$

$$v(t) = K_2 e^{-\alpha_2 t}\cos(\beta_2 t + \theta_2)$$

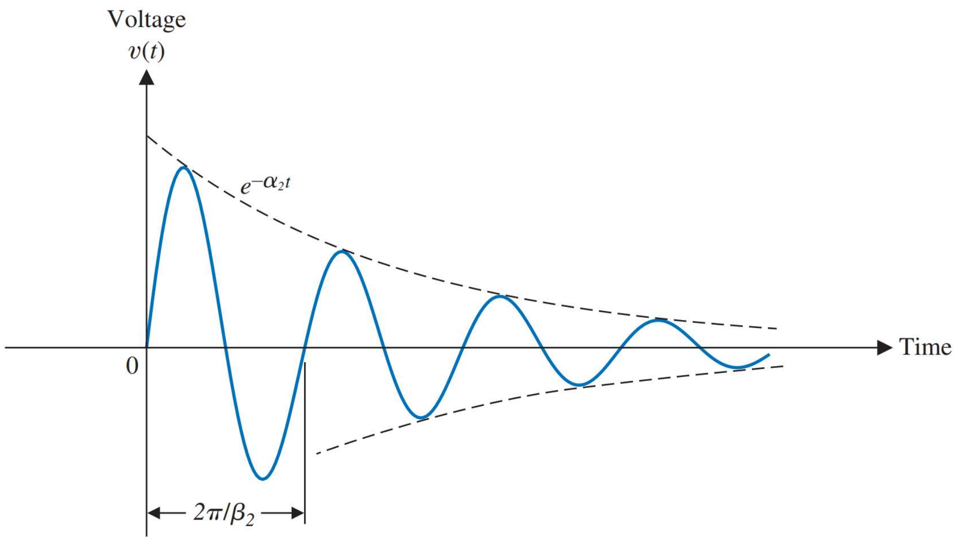


FIGURE 2.4
Typical voltage response for an *RLC* circuit.

Spring mass damper

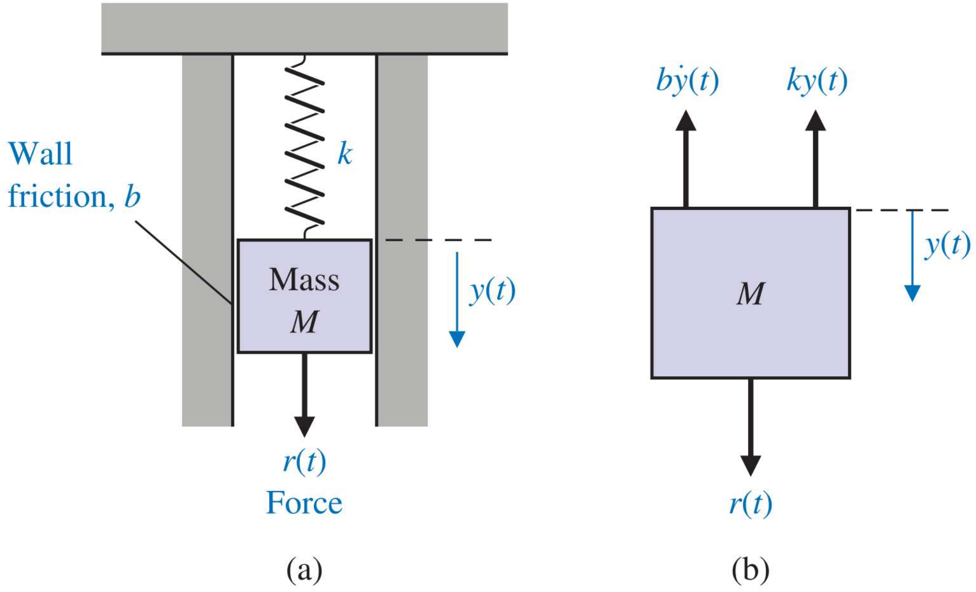


FIGURE 2.2
(a) Spring-mass-damper system.
(b) Free-body diagram.

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

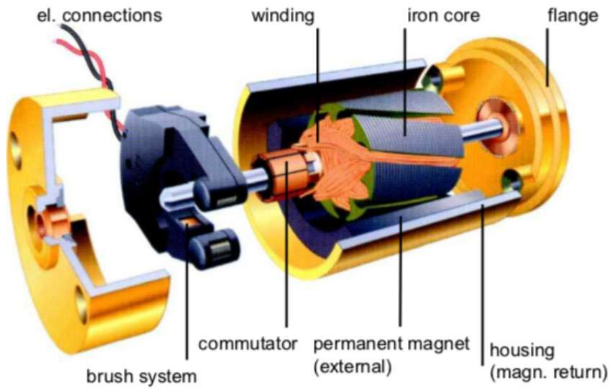
$$y(t) = K_1 e^{-\alpha_1 t}\sin(\beta_1 t + \theta_1)$$

Eletro-mechanical system

- Newtonian mechanics
- Faraday's law of induction
  - With a little bit of math of total differential and derivative laws
- Ampere force

$$J\frac{d^2}{dt^2}\Theta = D\frac{d}{dt}\Theta + K_T i_1$$

$$u_1 = Ri_1 + K_E\frac{d}{dt}\Theta + L_1\frac{d}{dt}i_1$$



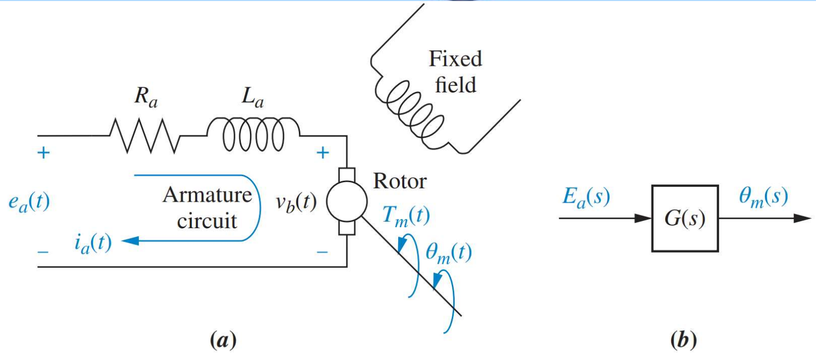


FIGURE 2.35 DC motor: **a.** schematic;[12] **b.** block diagram

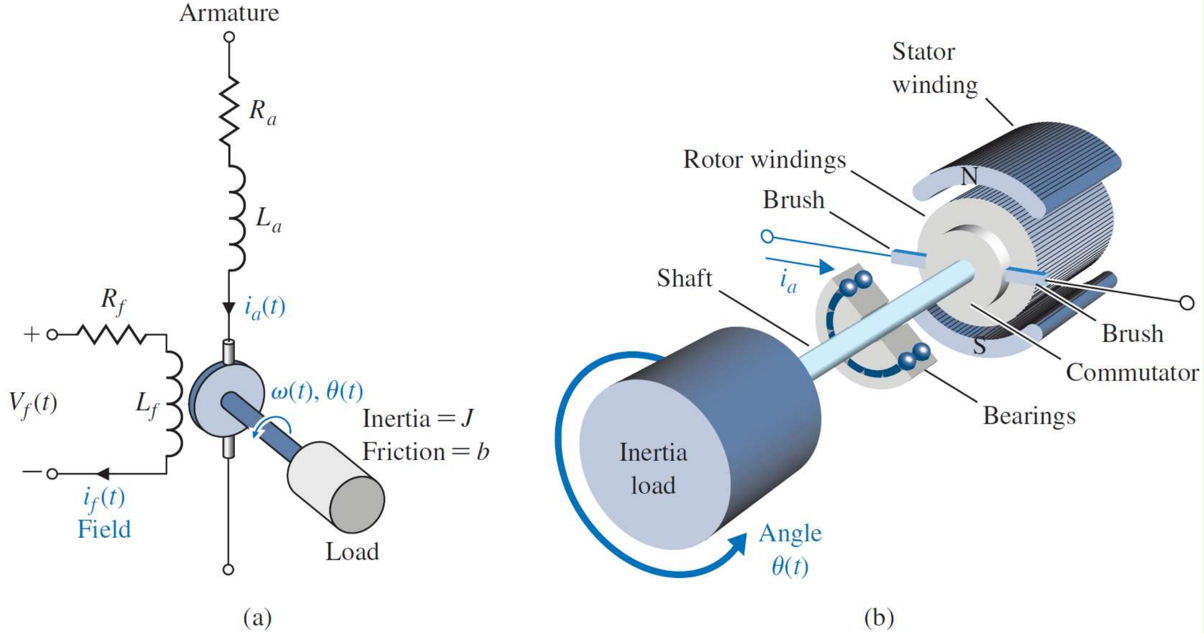
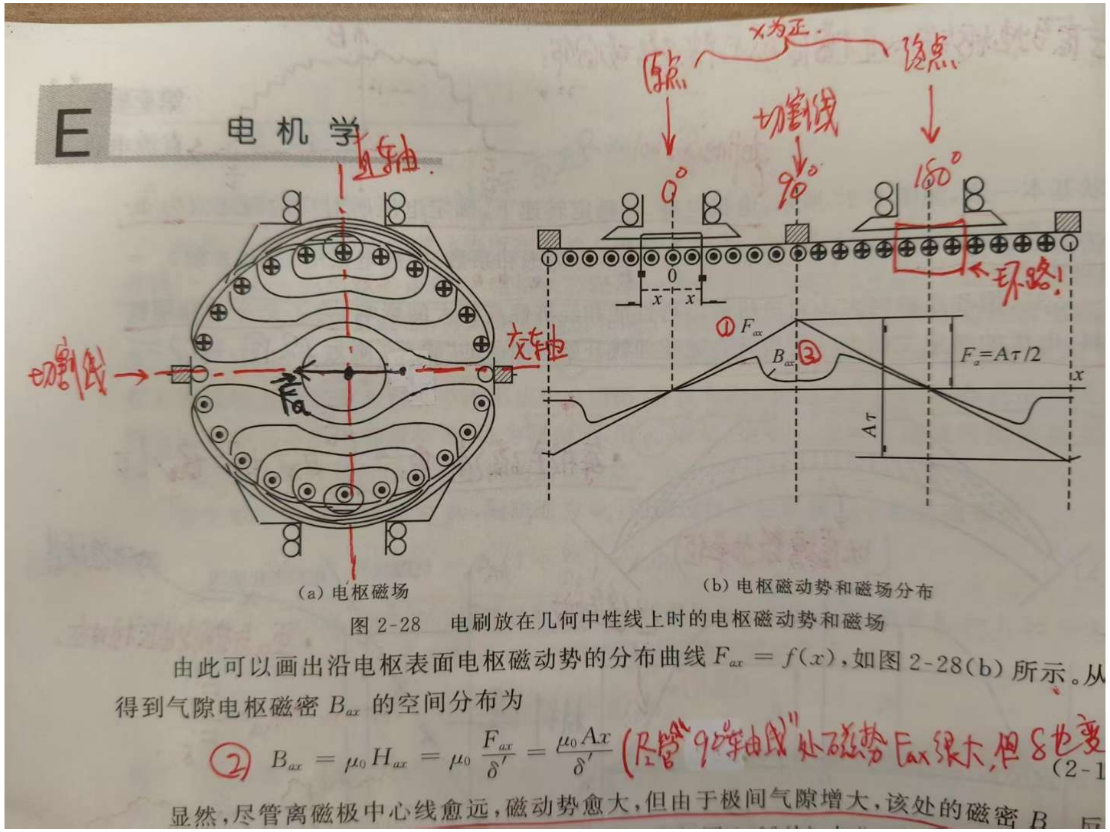
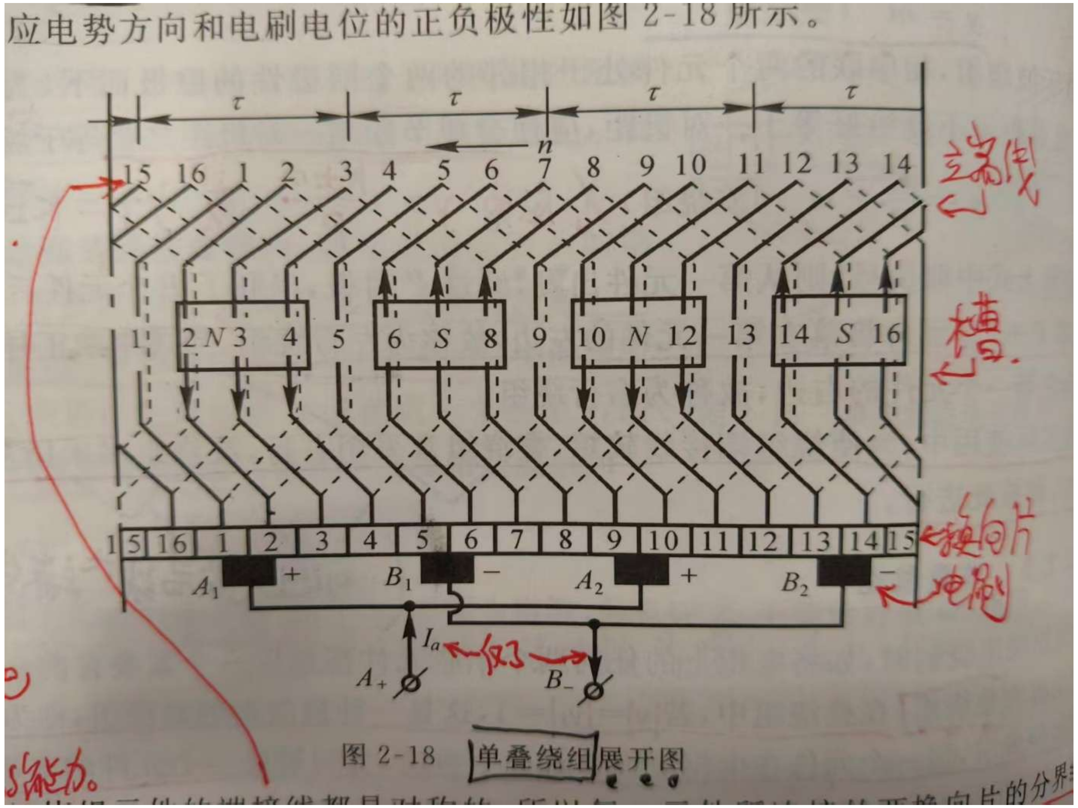


FIGURE 2.18
A DC motor
(a) electrical diagram and
(b) sketch.



E 电机学
(a) 电枢磁场
(b) 电枢磁动势和磁场分布
图 2-28 电刷放在几何中性线上时的电枢磁动势和磁场
由此可以画出沿电枢表面电枢磁动势的分布曲线 $F_{ax} = f(x)$，如图 2-28(b) 所示。从
得到气隙电枢磁密 B_{ax} 的空间分布为
$B_{ax} = \mu_0 H_{ax} = \mu_0 \frac{F_{ax}}{\delta'} = \frac{\mu_0 A x}{\delta'}$
显然，尽管离磁极中心线愈远，磁动势愈大，但由于极间气隙增大，该处的磁密
$F_a = A\tau/2$
$A\tau$
直轴
交轴
切割线
0°
90°
180°
环路！



应电势方向和电刷电位的正负极性如图 2-18 所示。
τ
n
端线
槽
换向片
电刷
A_1 B_1 A_2 B_2
I_a
A_+ B_-
图 2-18 单叠绕组展开图

《电机学》，浙江大学出版社

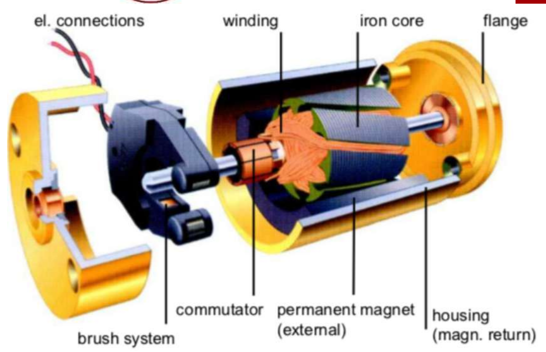
Mathematatic Model Overview

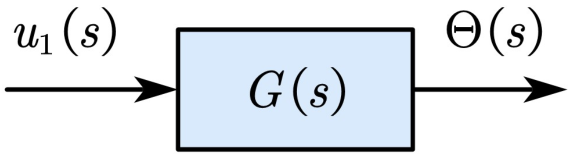
Overview of the Math Models

- O.D.E (from physics laws)
- Block diagram
- Transfer function (via Laplace transform)
- State space model
- Pole zero map (related to eigenvalues)
  - Root locus and pole placement

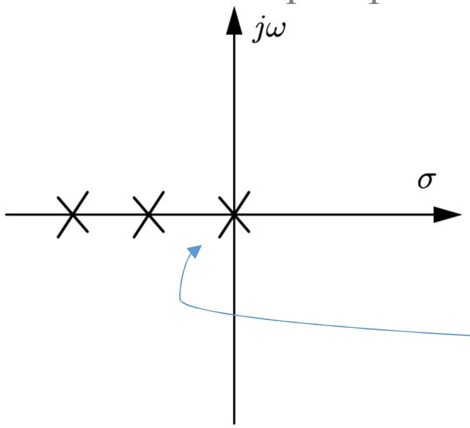
$$J\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Theta = D\frac{\mathrm{d}}{\mathrm{d}t}\Theta + K_T i_1$$

$$u_1 = Ri_1 + K_E\frac{\mathrm{d}}{\mathrm{d}t}\Theta + L_1\frac{\mathrm{d}}{\mathrm{d}t}i_1$$





Can you give details within $G(s)$?



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & K_T J^{-1}\\ 0 & -K_E L^{-1} & -RL^{-1}\end{bmatrix}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix} + \begin{bmatrix}0\\ 0\\ L^{-1}\end{bmatrix}u + \begin{bmatrix}0\\ J^{-1}\\ 0\end{bmatrix}\tau_L$$

Eigenvalues
(when input is set to 0)

Exercise for modeling of an e.m. system

Example 2.23

Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(a) and (b), find the transfer function, $\theta_L(s)/E_a(s)$.

SOLUTION: Begin by finding the mechanical constants, J_m and D_m, in Eq. (2.153). From Eq. (2.155), the total inertia at the armature of the motor is

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \qquad (2.164)$$

and the total damping at the armature of the motor is

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10 \qquad (2.165)$$

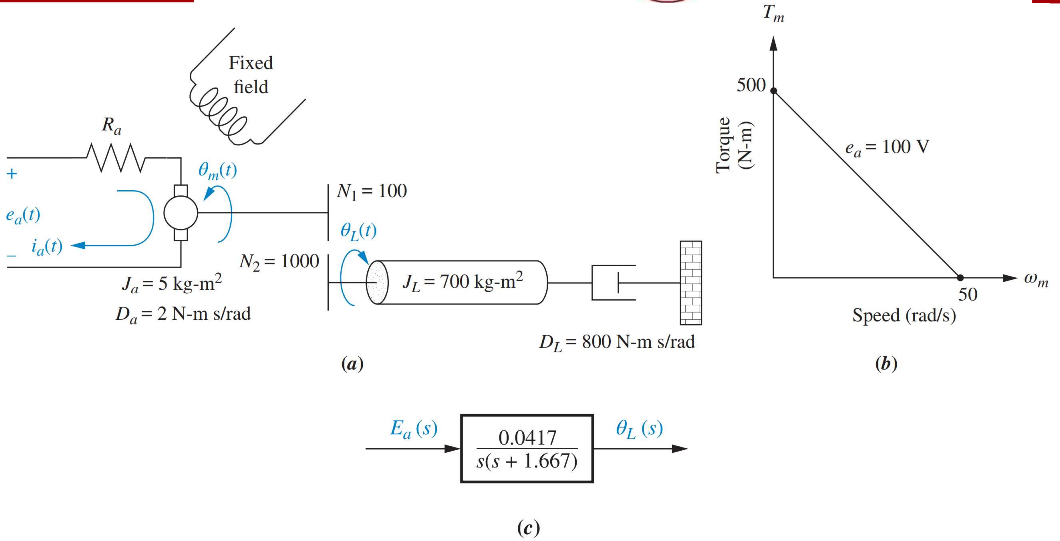
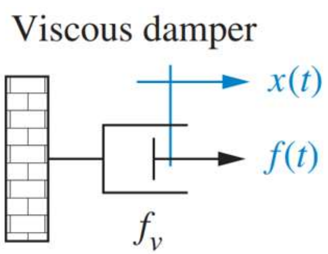


FIGURE 2.39 **a.** DC motor and load; **b.** torque-speed curve; **c.** block diagram

Generalizing the results, we can make the following statement: *Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio*

$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

where the impedance to be reflected is attached to the source shaft and is being reflected to the destination shaft. The next example demonstrates the application of the concept of reflected impedances as we find the transfer function of a rotational mechanical system with gears.



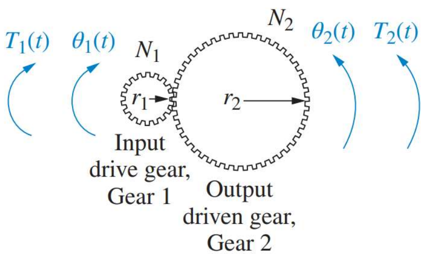


FIGURE 2.27 A gear system

State Space Model

State space model

- The simultaneous **first-order differential** equations about the state variables
  - State variables must be linearly independent; that is, no state variable can be written as a linear combination of the other state variables.
- **Algebraic** output equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

- E.g.,:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & K_T J^{-1}\\ 0 & -K_E L^{-1} & -RL^{-1}\end{bmatrix}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix}+\begin{bmatrix}0\\ 0\\ L^{-1}\end{bmatrix}u+\begin{bmatrix}0\\ J^{-1}\\ 0\end{bmatrix}\tau_L$$

$$y=\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}\Theta\\ \Omega\\ i\end{bmatrix}$$

- State space: The n-dimensional space whose axes are the state variables.

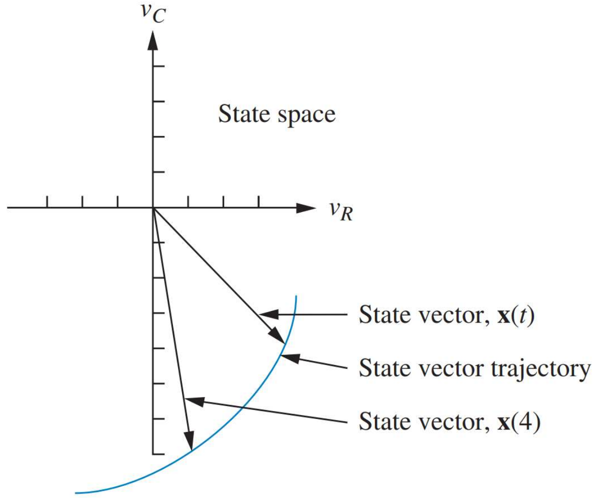


FIGURE 3.3 Graphic representation of state space and a state vector

State space model

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PROBLEM: Find the state-space representation of the electrical network shown in Figure 3.8. The output is $v_o(t)$.

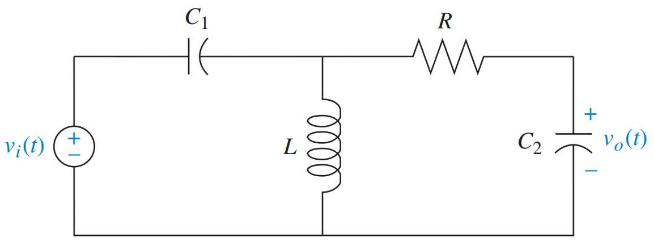

FIGURE 3.8 Electric circuit for Skill-Assessment Exercise 3.1

ANSWER:

PROBLEM: Represent the translational mechanical system shown in Figure 3.9 in state space, where $x_3(t)$ is the output.

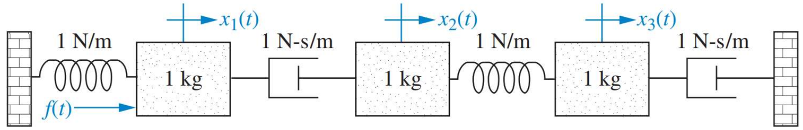

FIGURE 3.9 Translational mechanical system for Skill-Assessment Exercise 3.2

ANSWER:

From O.D.E to State space model

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At first we select a set of state variables, called *phase variables*, where each subsequent state variable is defined to be the derivative of the previous state variable.

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

$$x_1 = y,\quad x_2 = \frac{dy}{dt},\quad x_3 = \frac{d^2y}{dt^2},\quad \vdots \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

$$\dot{x}_1 = \frac{dy}{dt},\quad \dot{x}_2 = \frac{d^2y}{dt^2},\quad \dot{x}_3 = \frac{d^3y}{dt^3},\quad \vdots \quad \dot{x}_n = \frac{d^n y}{dt^n}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0x_1 - a_1x_2 \cdots - a_{n-1}x_n + b_0 u \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u \quad (3.52)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Converting a Transfer Function with a Constant Term in the Numerator

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

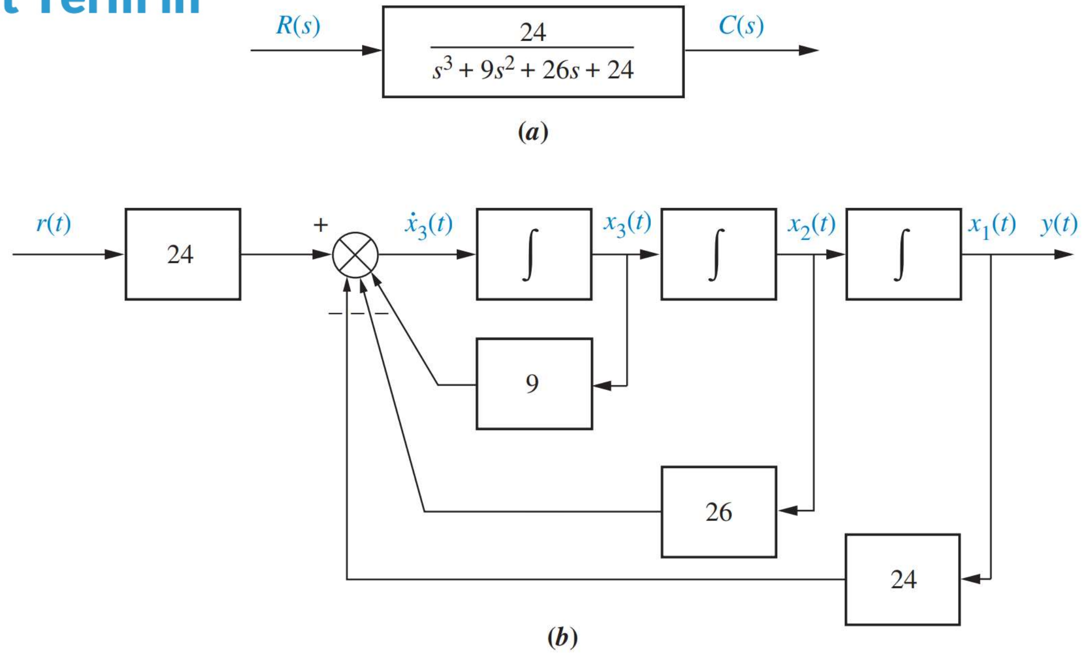

FIGURE 3.10 **a.** Transfer function; **b.** equivalent block diagram showing phase variables. Note: $y(t) = c(t)$.

From transfer function to state space model (2)

Converting a Transfer Function with a Polynomial in the Numerator

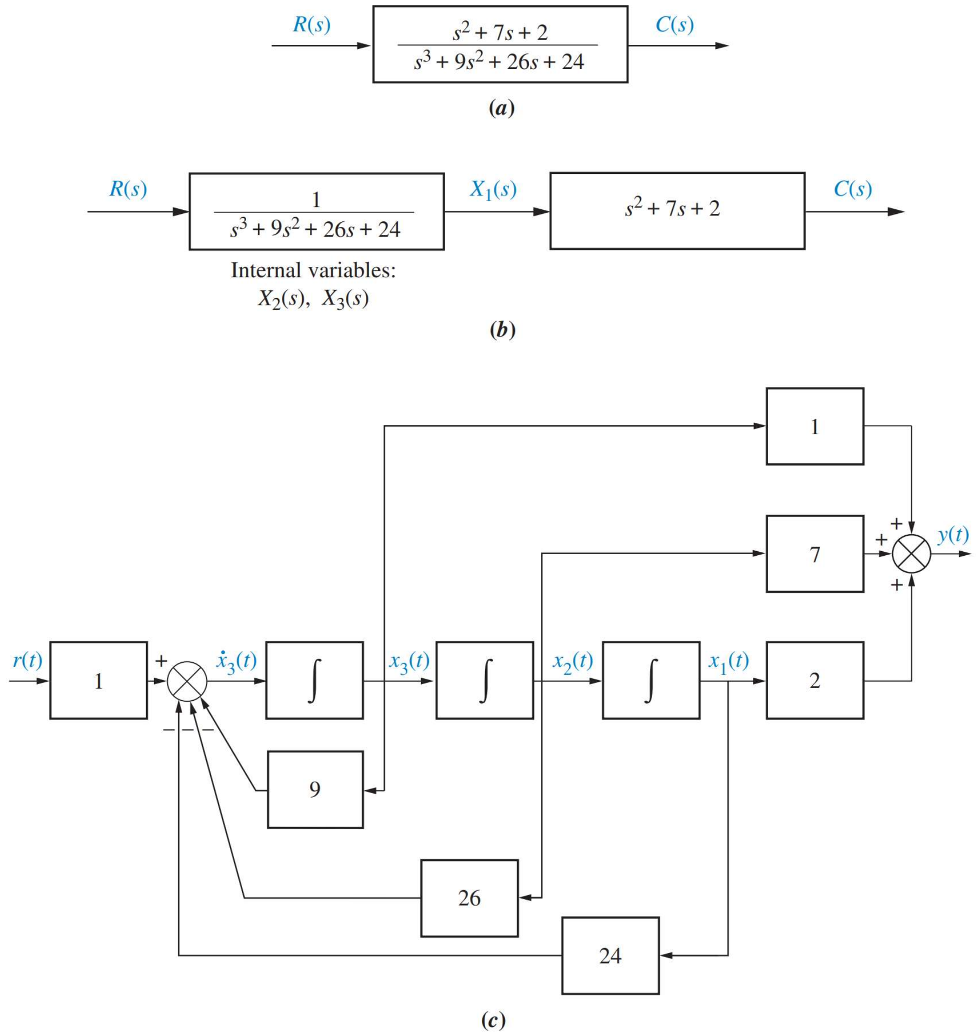

FIGURE 3.12 **a.** Transfer function; **b.** decomposed transfer function; **c.** equivalent block diagram Note: $y(t) = c(t)$.

From transfer function to state space model (2)

PROBLEM: Find the state equations and output equation for the phase-variable representation of the transfer function $G(s) = \dfrac{2s + 1}{s^2 + 7s + 9}$.

ANSWER:

State space model to transfer funtion (ss2tf)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

Transfer function matrix

Single input single output

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

State space model to transfer funtion (ss2tf)

PROBLEM: Given the system defined by Eq. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u \qquad (3.74a)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x} \qquad (3.74b)$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3+3s^2+2s+1}$$

$$(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

$$T(s) = \frac{10(s^2+3s+2)}{s^3+3s^2+2s+1}$$

https://en.wikipedia.org/wiki/Adjugate_matrix#:~:text=In%20linear%20algebra%2C%20the%20adjugate,matrix%20is%20the%20conjugate%20transpose.

State space model to transfer funtion (ss2tf)

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TryIt 3.2

Use the following MATLAB and the Control System Toolbox statements to obtain the transfer function shown in Skill-Assessment Exercise 3.4 from the state-space representation of Eq. (3.78).

```
A=[-4 -1.5; 4 0];
B=[2 0]';
C=[1.5 0.625];
D=0;
T=ss(A,B,C,D);
T=tf(T)
```

PROBLEM: Convert the state and output equations shown in Eq. (3.78) to a transfer function.

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t) \qquad (3.78a)$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix}\mathbf{x} \qquad (3.78b)$$

ANSWER:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathrm{adj}(\mathbf{A}) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

State space model is not unique

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u}$$

PROBLEM: Given the system represented in state space by Eqs. (5.73),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \qquad (5.73a)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x} \qquad (5.73b)$$

transform the system to a new set of state variables, $\mathbf{z}$, where the new state variables are related to the original state variables, $\mathbf{x}$, as follows:

$$z_1 = 2x_1 \qquad (5.74a)$$

$$z_2 = 3x_1 + 2x_2 \qquad (5.74b)$$

$$z_3 = x_1 + 4x_2 + 5x_3 \qquad (5.74c)$$

$$\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\mathbf{x} = \mathbf{P}^{-1}\mathbf{x}$$

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix}\begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

Diagonal state space representation

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Eigenvector. The eigenvectors of the matrix $\mathbf{A}$ are all vectors, $\mathbf{x_i} \neq \mathbf{0}$, which under the transformation $\mathbf{A}$ become multiples of themselves; that is,

$$\mathbf{A}\mathbf{x_i} = \lambda_i \mathbf{x_i} \tag{5.80}$$

where λ_i's are constants.

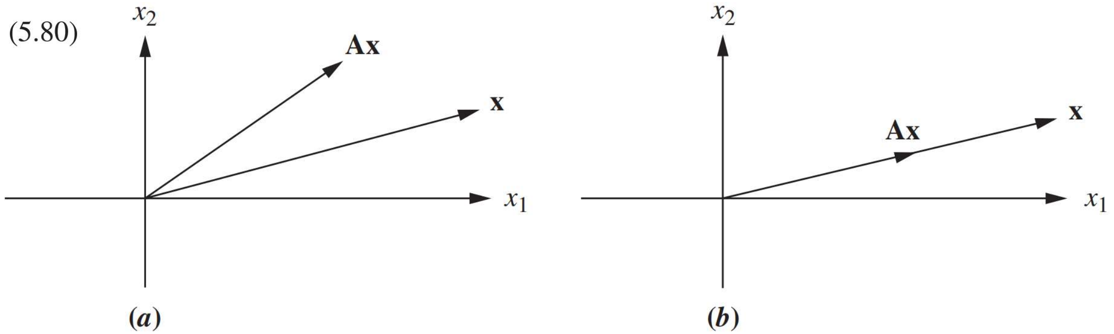

FIGURE 5.32 To be an eigenvector, the transformation $\mathbf{Ax}$ must be collinear with $\mathbf{x}$; thus, in (*a*), $\mathbf{x}$ is not an eigenvector; in (*b*), it is.

Eigenvalue. The eigenvalues of the matrix $\mathbf{A}$ are the values of λ_i that satisfy Eq. (5.80) for $\mathbf{x_i} \neq \mathbf{0}$.

To find the eigenvectors, we rearrange Eq. (5.80). Eigenvectors, $\mathbf{x_i}$, satisfy

$$\mathbf{0} = (\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x_i} \tag{5.81}$$

Solving for $\mathbf{x_i}$ by premultiplying both sides by $(\lambda_i \mathbf{I} - \mathbf{A})^{-1}$ yields

$$\mathbf{x_i} = (\lambda_i \mathbf{I} - \mathbf{A})^{-1}\mathbf{0} = \frac{\text{adj}(\lambda_i \mathbf{I} - \mathbf{A})}{\det(\lambda_i \mathbf{I} - \mathbf{A})}\mathbf{0} \tag{5.82}$$

Since $\mathbf{x_i} \neq \mathbf{0}$, a nonzero solution exists if

$$\det(\lambda_i \mathbf{I} - \mathbf{A}) = 0 \tag{5.83}$$

from which λ_i, the eigenvalues, can be found.

- Diagonal representation
  - Equivalent to partial fraction expansion of t.f.
  - Use eigenvectors to form a transform matrix $\mathbf{P}$

$$\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x_n}]$$

$$\mathbf{A}\mathbf{x_i} = \lambda_i \mathbf{x_i}$$

$$\mathbf{AP} = \mathbf{PD}$$

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$$

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu}$$

$$\mathbf{y} = \mathbf{CPz} + \mathbf{Du}$$

Left $\mathbf{D}$ is diagonal matrix with eigenvalues, while right $\mathbf{D}$ is the through-input matrix, often $\mathbf{D} = 0$

Diagonal state space representation

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Diagonalizing a System in State Space

PROBLEM: Given the system of Eqs. (5.94), find the diagonal system that is similar.

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u \qquad (5.94a)$$

$$y = [2 \quad 3]\,\mathbf{x} \qquad (5.94b)$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\mathbf{CP} = [2 \quad 3]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [5 \quad -1]$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}u$$

$$y = [5 \quad -1]\,\mathbf{z}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu}$$

$$\mathbf{y} = \mathbf{CPz} + \mathbf{Du}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| = \begin{vmatrix} \lambda+3 & -1 \\ -1 & \lambda+3 \end{vmatrix} = \lambda^2 + 6\lambda + 8$$

$$\mathbf{A}\mathbf{x_i} = \lambda\mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Diagonal state space representation

PROBLEM: For the system represented in state space as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}u$$
$$y = \begin{bmatrix} 1 & 4 \end{bmatrix}\mathbf{x}$$

convert the system to one where the new state vector, $\mathbf{z}$, is

$$\mathbf{z} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\mathbf{x}$$

ANSWER:

$$\dot{\mathbf{z}} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -3 \\ -11 \end{bmatrix}u$$
$$y = \begin{bmatrix} 0.8 & -1.4 \end{bmatrix}\mathbf{z}$$

PROBLEM: For the original system find the diagonal system that is similar.

ANSWER:

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}u$$
$$y = \begin{bmatrix} -2.121 & 2.6 \end{bmatrix}\mathbf{z}$$

Solution of the State Equations

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$= \frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathscr{L}^{-1}\left[\frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right] = \boldsymbol{\Phi}(t)$$

Solution of the State Equations

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

for scalar case

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s).$$

$$x(t) = e^{at}x(0) + \int_0^t e^{+a(t-\tau)}bu(\tau)d\tau.$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$
$$= \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)]$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathscr{L}^{-1}\left[\frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\right] = \mathbf{\Phi}(t)$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau$$

zero-input response

Zero-state response as convolution integral

*The **matrix exponential** function is defined by in a similar Taylor series form*

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots + \frac{\mathbf{A}^kt^k}{k!} + \cdots,$$

Block Diagram

Block diagram

- Basic components:

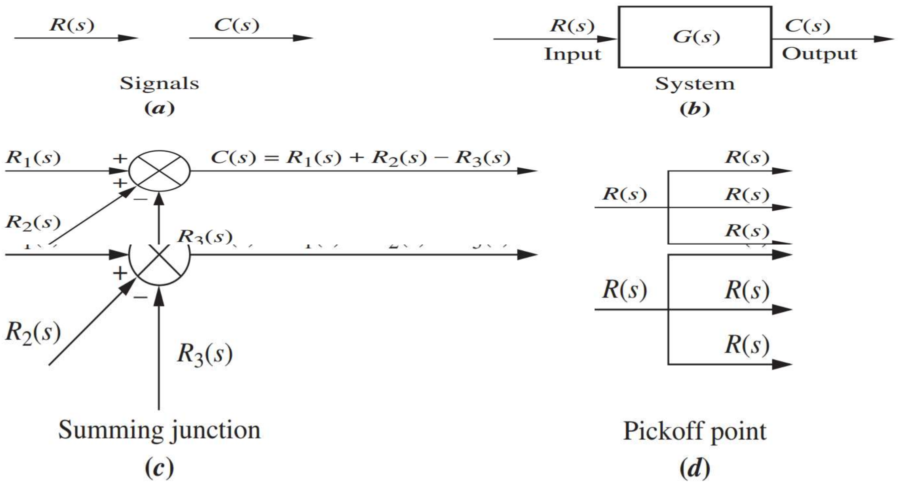

FIGURE 5.2 Components of a block diagram for a linear, time-invariant system

- Cascade and parallel:

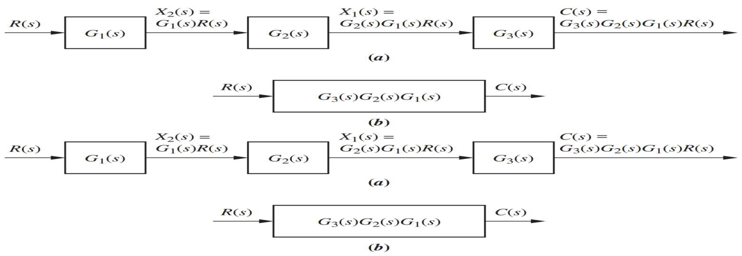

FIGURE 5.3 a. Cascaded subsystems; b. equivalent transfer function

FIGURE 5.3 a. Cascaded subsystems; b. equivalent transfer function

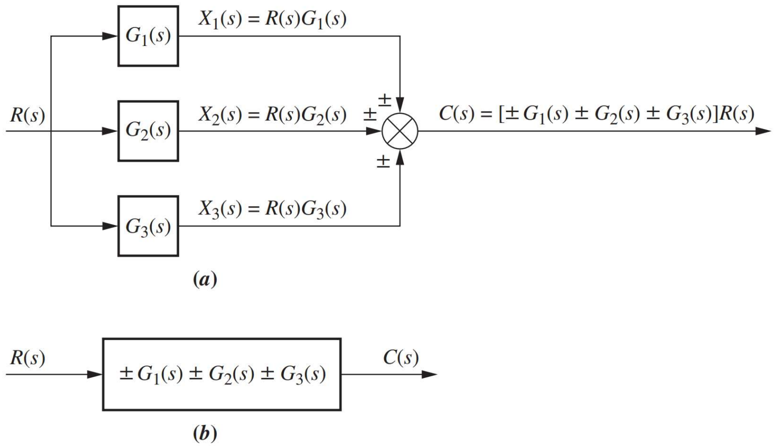

FIGURE 5.5 a. Parallel subsystems; b. equivalent transfer function

Block diagram reduction

- Feedback

$$G_e(s) = \frac{G(s)}{1 \pm G(s)H(s)}$$

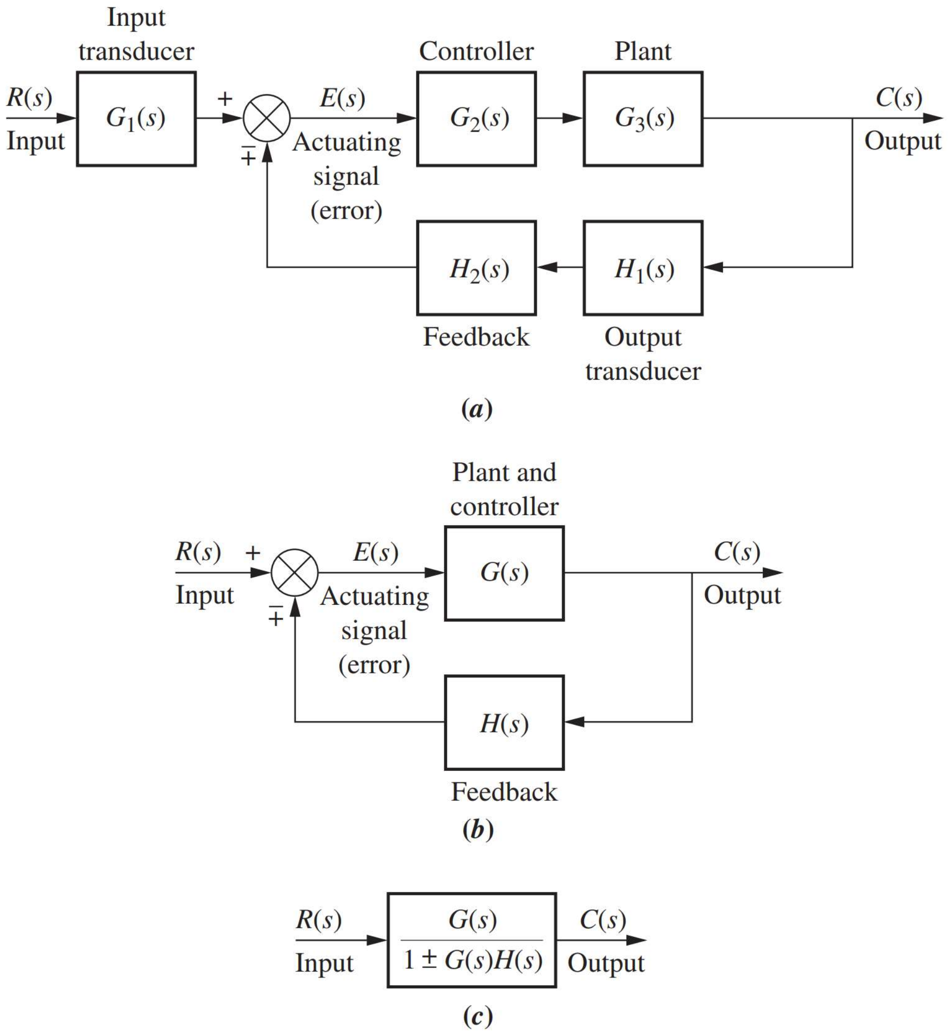

FIGURE 5.6 **a.** Feedback control system; **b.** simplified model; **c.** equivalent transfer function

Block diagram reduction

- Moving blocks

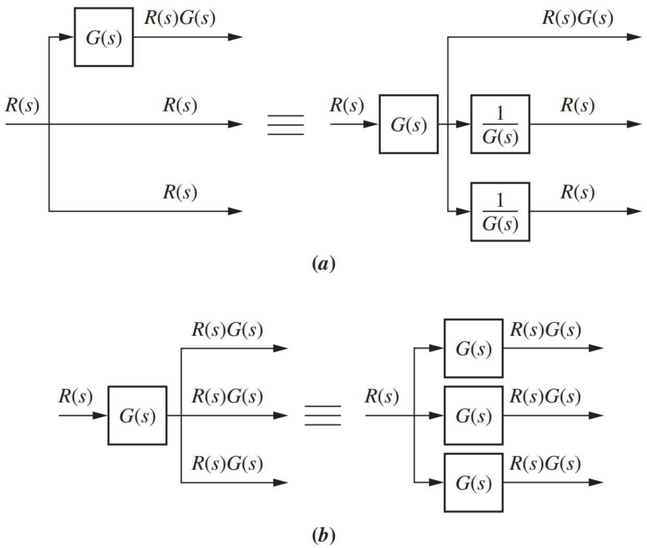

FIGURE 5.8 Block diagram algebra for pickoff points—equivalent forms for moving a block **a.** to the left past a pickoff point; **b.** to the right past a pickoff point

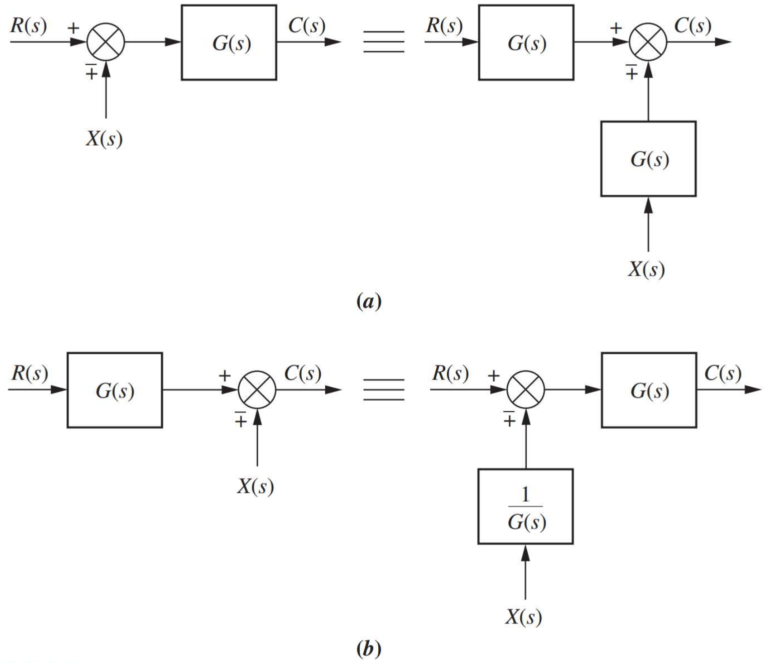

FIGURE 5.7 Block diagram algebra for summing junctions—equivalent forms for moving a block **a.** to the left past a summing junction; **b.** to the right past a summing junction

Block diagram reduction

PROBLEM: Reduce the block diagram shown in Figure 5.9 to a single transfer function.

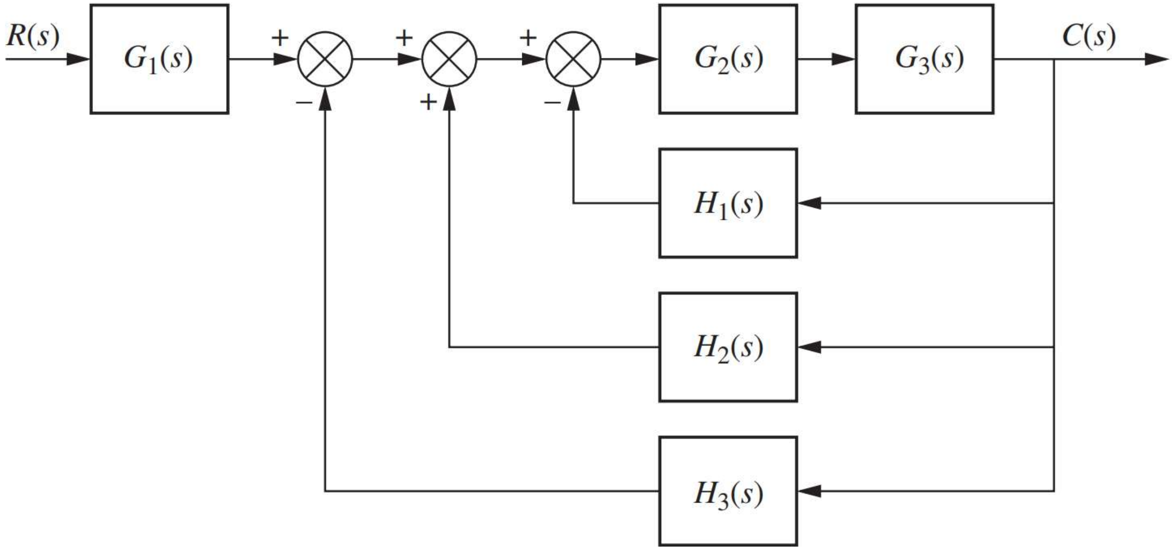

FIGURE 5.9 Block diagram for Example 5.1

Block diagram reduction

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ANSWER:

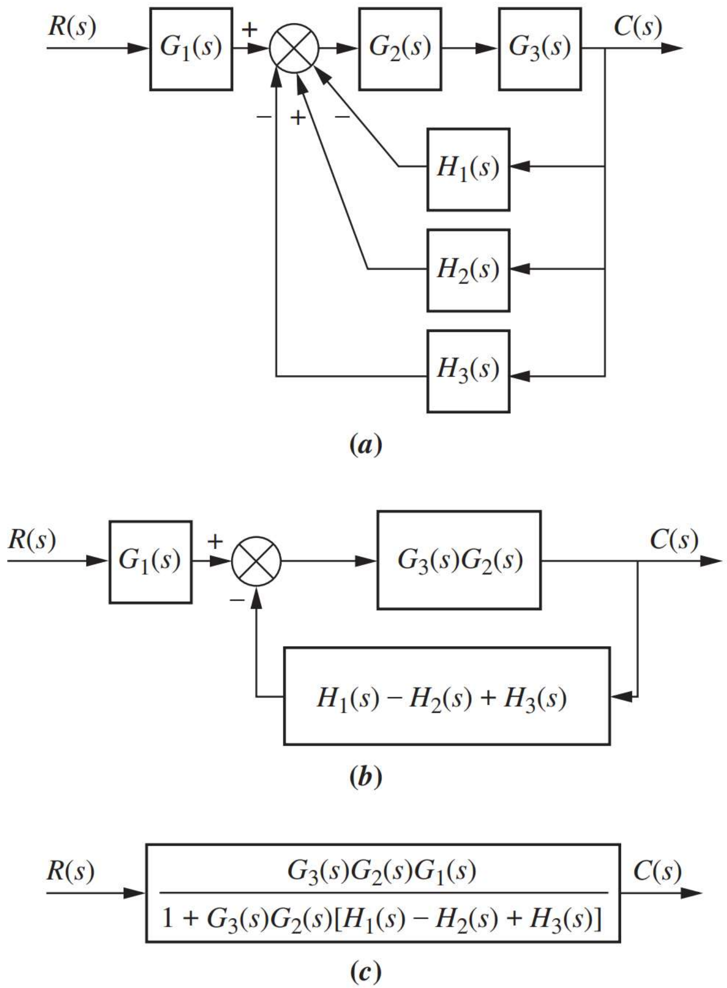

FIGURE 5.10 Steps in solving Example 5.1: **a.** Collapse summing junctions; **b.** form equivalent cascaded system in the forward path and equivalent parallel system in the feedback path; **c.** form equivalent feedback system and multiply by cascaded $G_1(s)$

Block Diagram Reduction by Moving Blocks

PROBLEM: Reduce the system shown in Figure 5.11 to a single transfer function.

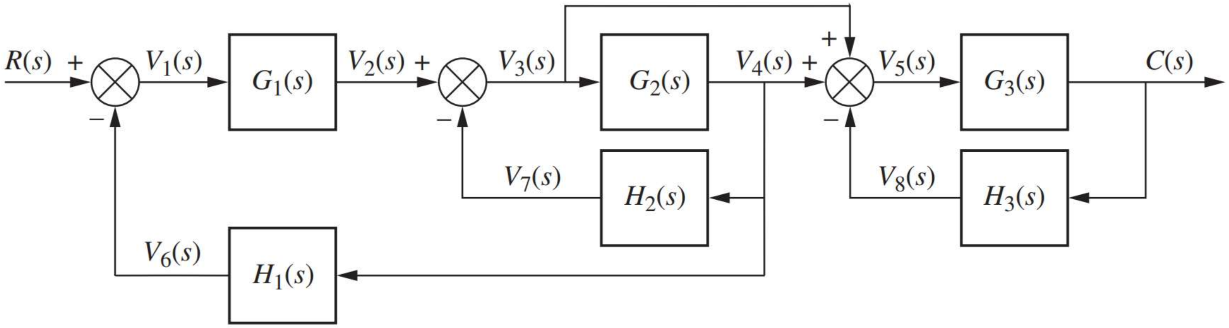

FIGURE 5.11 Block diagram for Example 5.2

Block diagram reduction

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ANSWER:

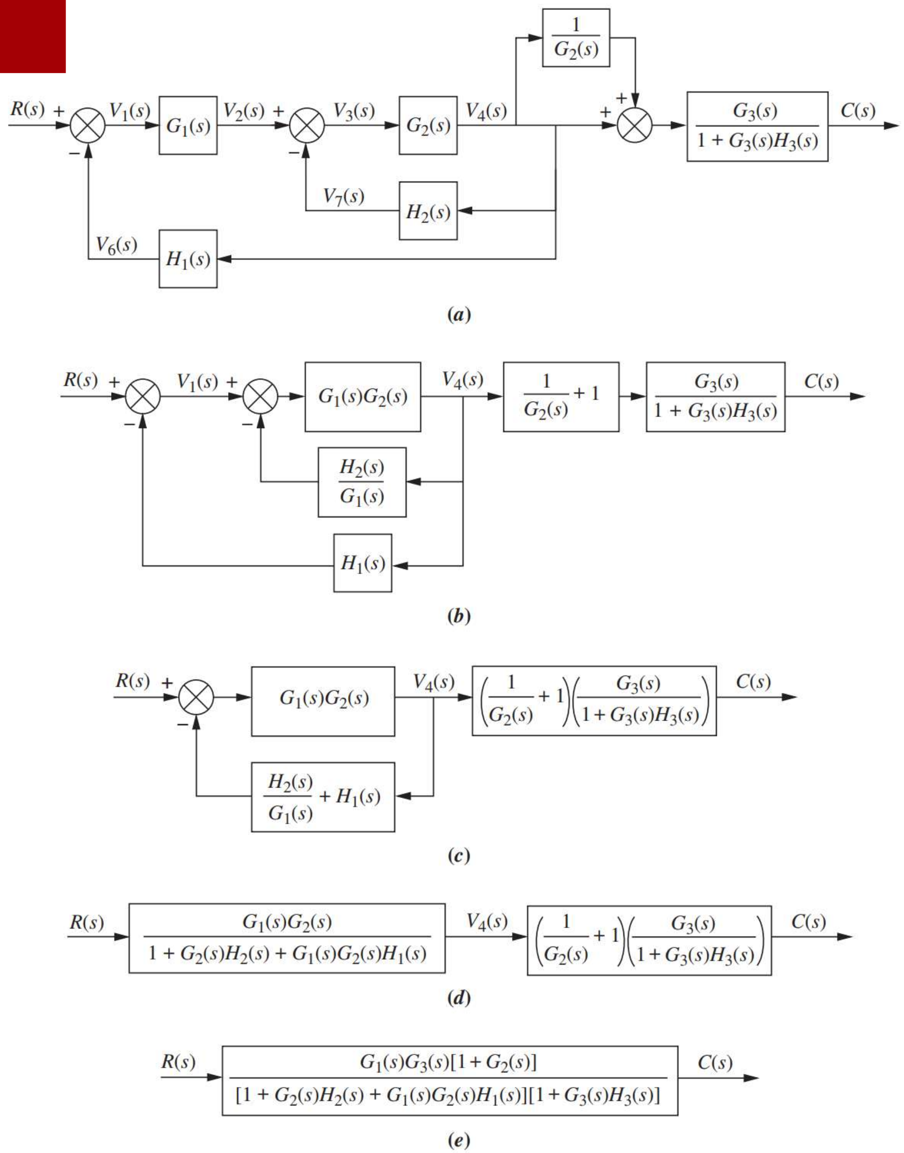

FIGURE 5.12 Steps in the block diagram reduction for Example 5.2

Block diagram reduction

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PROBLEM: Find the equivalent transfer function, $T(s) = C(s)/R(s)$, for the system shown in Figure 5.13.

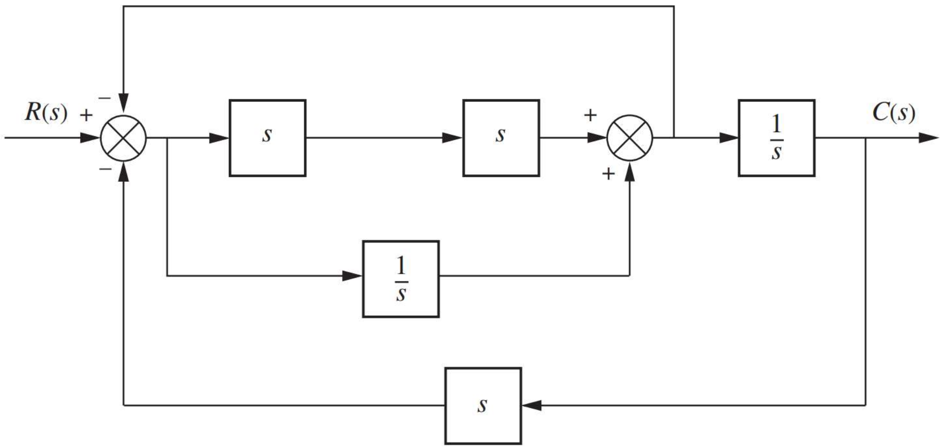


FIGURE 5.13 Block diagram for Skill-Assessment Exercise 5.1

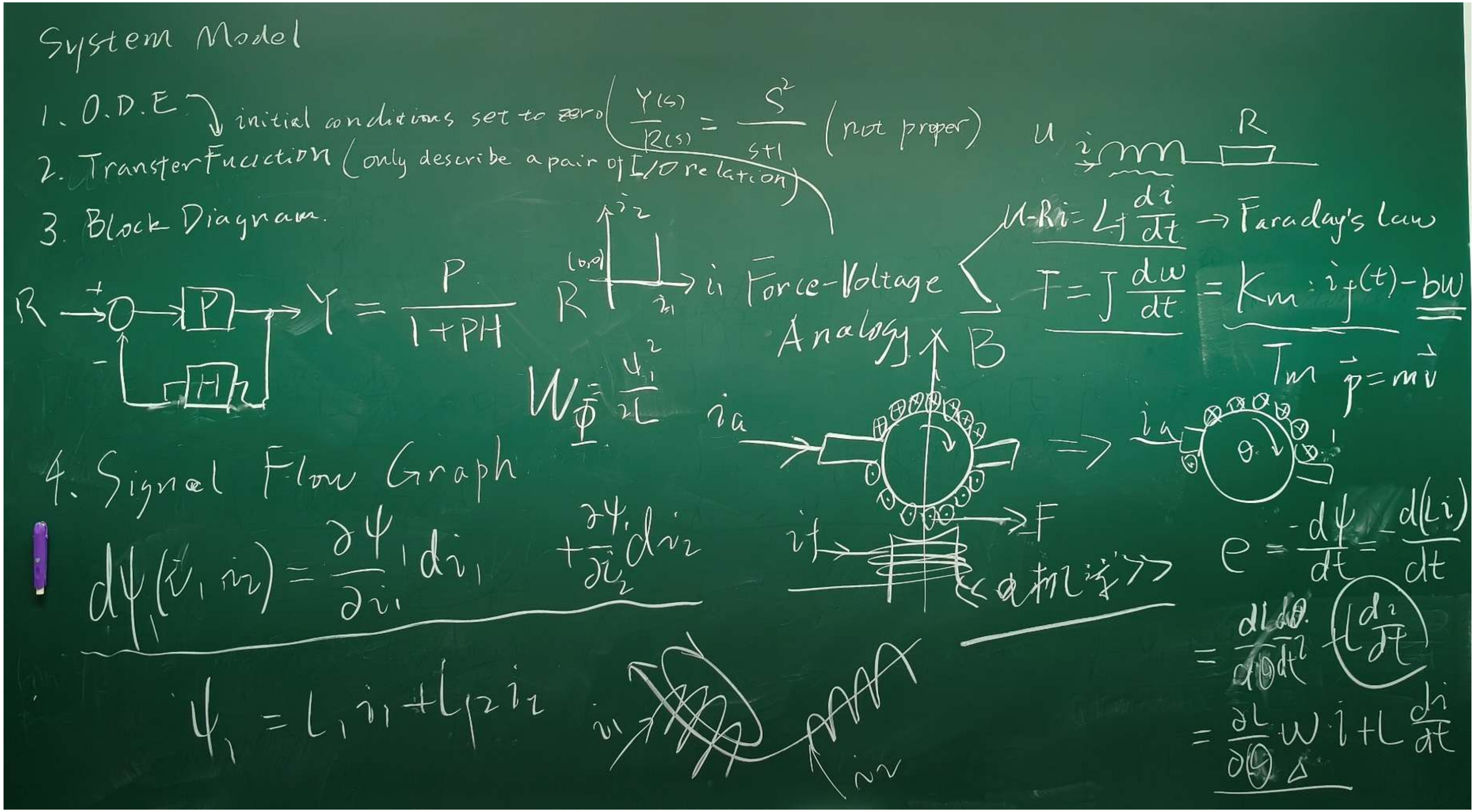
ANSWER:

TryIt 5.1

Use the following MATLAB and Control System Toolbox statements to find the closed-loop transfer function of the system in Example 5.2 if all $G_i(s) = 1/(s+1)$ and all $H_i(s) = 1/s$.

```
G1=tf(1,[1 1]);
G2=G1; G3=G1;
H1=tf(1,[1 0]);
H2=H1; H3=H1;
System=append...
 (G1, G2, G3, H1, H2, H3);
input=1; output=3;
Q= [1  -4   0   0   0
    2   1  -5   0   0
    3   2   1  -5  -6
    4   2   0   0   0
    5   2   0   0   0
    6   3   0   0   0];
T=connect(System,...
 Q, input, output);
T=tf(T); T=minreal(T)
```

Appendix


System Model
1. O.D.E
initial conditions set to zero
2. Transfer Function (only describe a pair of I/O relation)
3. Block Diagram.
Y(s)/R(s) = s²/(s+1) (not proper)
R
P
H
Y = P/(1+PH)
Force-Voltage Analogy
Faraday's law
T = J dω/dt = Km · if(t) − bω
Tm
p = mv
B
F
4. Signal Flow Graph
e = −dψ/dt = −d(Li)/dt
ψ1 = L1 i1 + L12 i2

Solution of Differential Equations

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Conclusion: The solution of the state differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Proof:

- *Uniqueness of Solutions*

If we have two solutions $x_1, x_2 : \mathbb{R} \to \mathbb{R}^{n_x}$, then $y = x_1 - x_2$ satisfies

$$\dot{y}(t) = Ay(t) \quad \text{with} \quad y(0) = 0 .$$

The auxiliary function $v(t) = e^{-At}y(t)$ satisfies

$$\begin{aligned} \dot{v}(t) &= -Ae^{-At}y(t) + e^{-At}Ay(t) = -Ae^{-At}y(t) + Ae^{-At}y(t) = 0 \\ v(0) &= 0 , \end{aligned}$$

$$\Longrightarrow \quad v(t) = y(t) = 0 \quad \Longrightarrow \quad x_1 = x_2 .$$

Solution of Differential Equations

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Conclusion: The solution of the state differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\mathbf{B}\mathbf{u}(\tau)d\tau.$$

Proof:

- *Verify the ODE*

Generalized Leibniz integral rule.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} g(t,\tau)\,\mathrm{d}\tau = g(t,b(t))\,\dot{b}(t) - g(t,a(t))\,\dot{a}(t) + \int_{a(t)}^{b(t)} g_t(t,\tau)\,\mathrm{d}\tau\,.$$

$$\dot{x}(t) = A\,e^{At}x(0) + e^{A(t-t)}Bu(t) + \int_0^t A\,e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= A\left[e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] + Bu(t)$$

$$= Ax(t) + Bu(t)$$

Solution of Differential Equations

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Specially, the solution of an unforced system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

is found to be

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$$

The matrix exponential function describes the unforced response of the system and is called the fundamental or state transition matrix $\Phi(t, 0)$.

Thus, the general solution can be written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau.$$

NOTE, up to now, we are talking about LTI system, for nonlinear or time-varying, there is NO nice general solution form.