



# Lecture 7 - Phasor

A beginning of AC circuits

*AC = alternating current;*

Circuits driven by sinusoidal current or voltage sources are *AC circuits*



# Outline

- Sinusoidal signals
- Phasor



# Why Sinusoidal signals?

- Numbers of natural phenomenon are sinusoidal in nature.
  - Motion of a pendulum, vibration of a string, ripples on ocean surface
- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
  - In the late 1800's there was a battle between proponents of DC and AC. AC won out due to its efficiency for long distance transmission.
- Lastly, they are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.
- Through *Fourier analysis*, any practical periodic function can be represented as sum of sinusoids.



# Sinusoidal Signal (Current or Voltage)

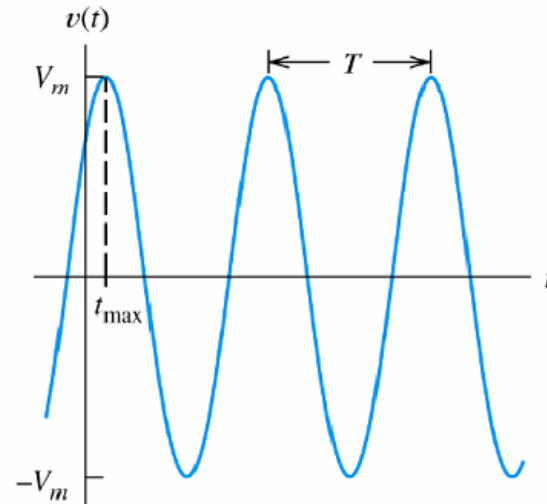
$$v(t) = V_m \cos(\omega t + \theta)$$

$V_m$  is the **peak value**

$\omega$  is the **angular frequency** in radians per second

$(\omega t + \theta)$  is the **phase angle**

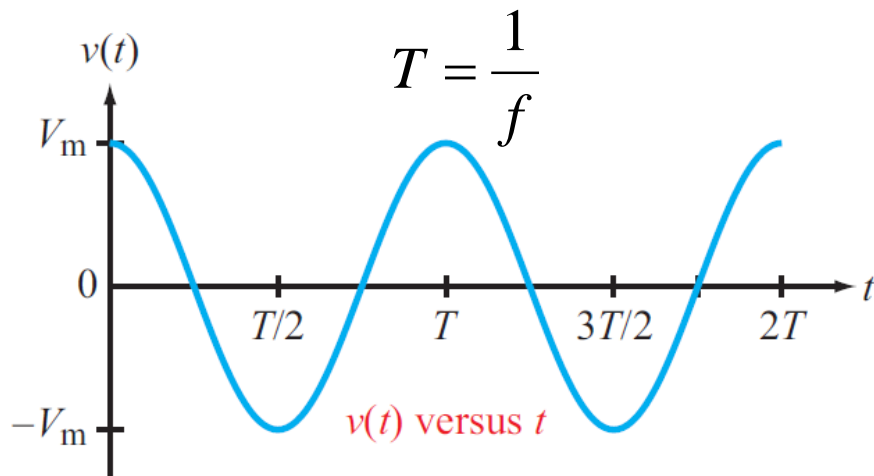
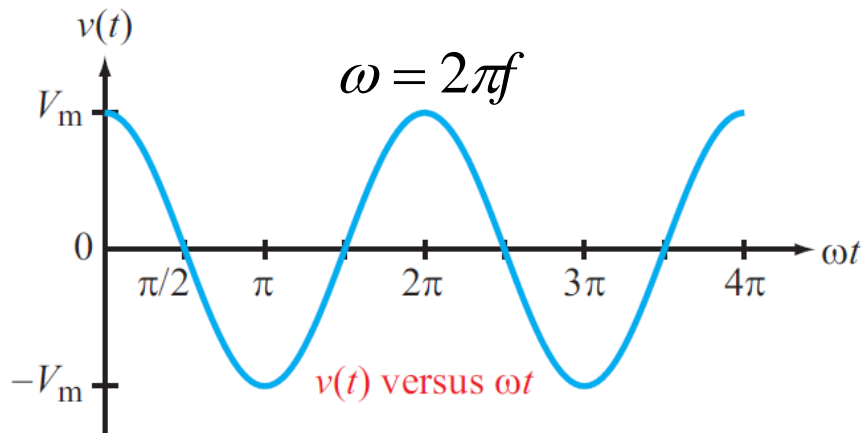
$$\omega = 2\pi f \quad f = \frac{1}{T}$$





# Sinusoidal Signals

$$v(t) = V_m \cos(\omega t + \theta)$$



## Useful relations

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$$\sin x = \pm \cos(x \mp 90^\circ)$$

$$\cos x = \pm \sin(x \pm 90^\circ)$$

$$\sin x = -\sin(x \pm 180^\circ)$$

$$\cos x = -\cos(x \pm 180^\circ)$$

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

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$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

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$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

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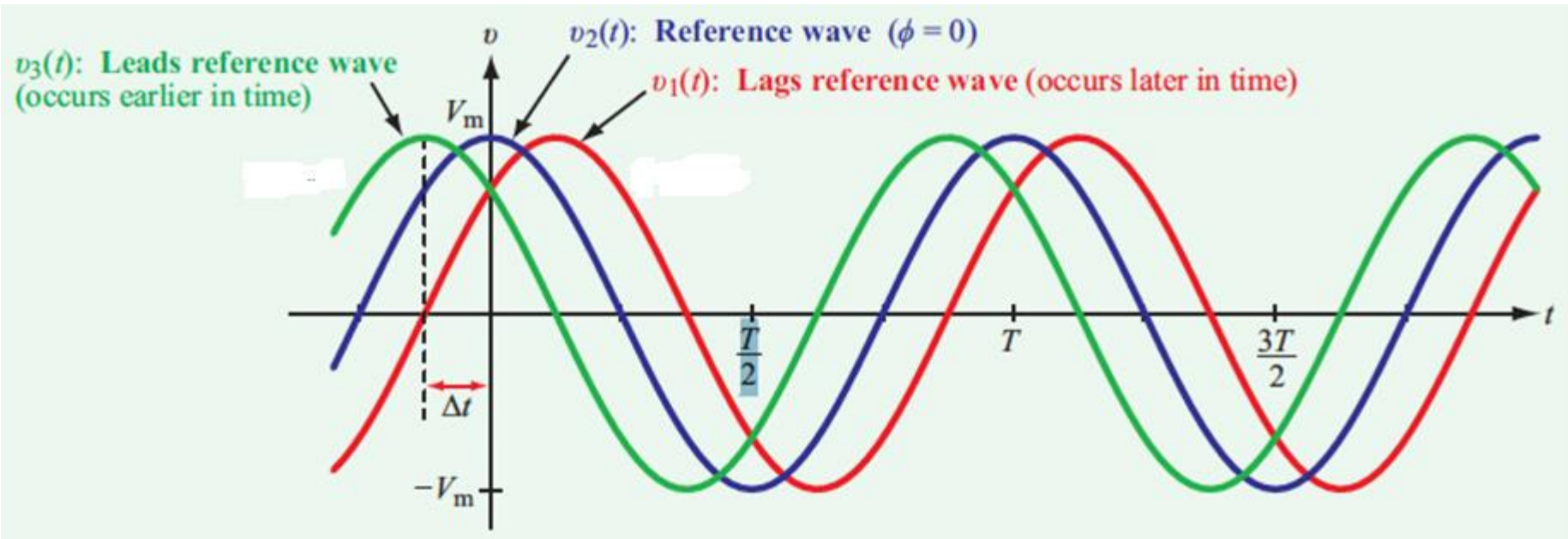


# Phase Lead/Lag

$$V_m \cos \frac{2\pi t}{T}$$

$$V_m \cos \left( \frac{2\pi t}{T} - \frac{\pi}{4} \right)$$

$$V_m \cos \left( \frac{2\pi t}{T} + \frac{\pi}{4} \right)$$



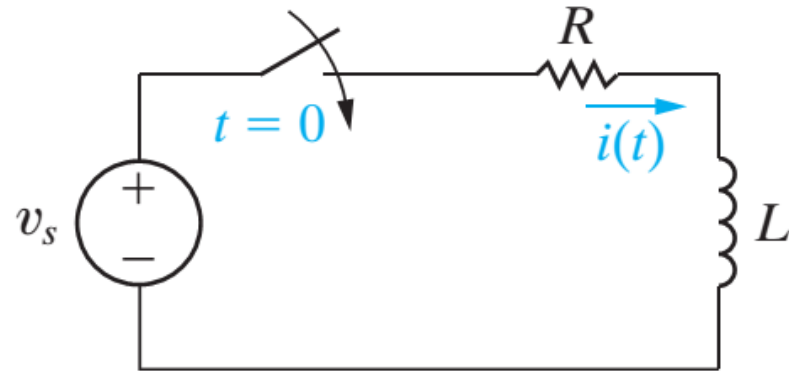


## The Sinusoidal Response

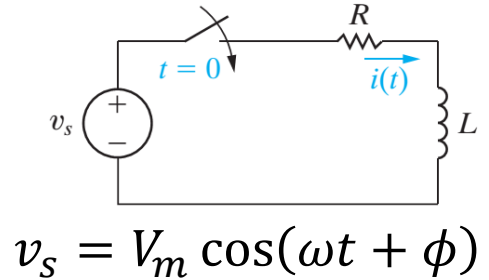
$$v_s = V_m \cos(\omega t + \phi), i(0^-) = 0.$$

Find  $i(t)$ ,  $t \geq 0$ .

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$



# Sinusoidal Steady-State Response



$$i = \underbrace{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}}_{\text{Transient response}} + \underbrace{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}_{\text{Steady-state response}}$$

- **Steady-state response** is sinusoidal
- Response frequency = source frequency
- **Magnitude & phase** (initial phase angle) of **S.S.** response differs from that of source



# Outline

- Sinusoidal signals
- Phasor



# Phasor

$$v(t) = V \cos(\omega t + \phi) = \operatorname{Re}\{V e^{j\phi} e^{j\omega t}\} = \operatorname{Re}(\mathbf{V} e^{j\omega t})$$

$$\mathbf{V} = V e^{j\phi}$$

- As Euler's formula:

$$e^{\pm j\phi} = \cos \phi \pm j \sin \phi$$

$$\cos \phi = \operatorname{Re}(e^{j\phi})$$

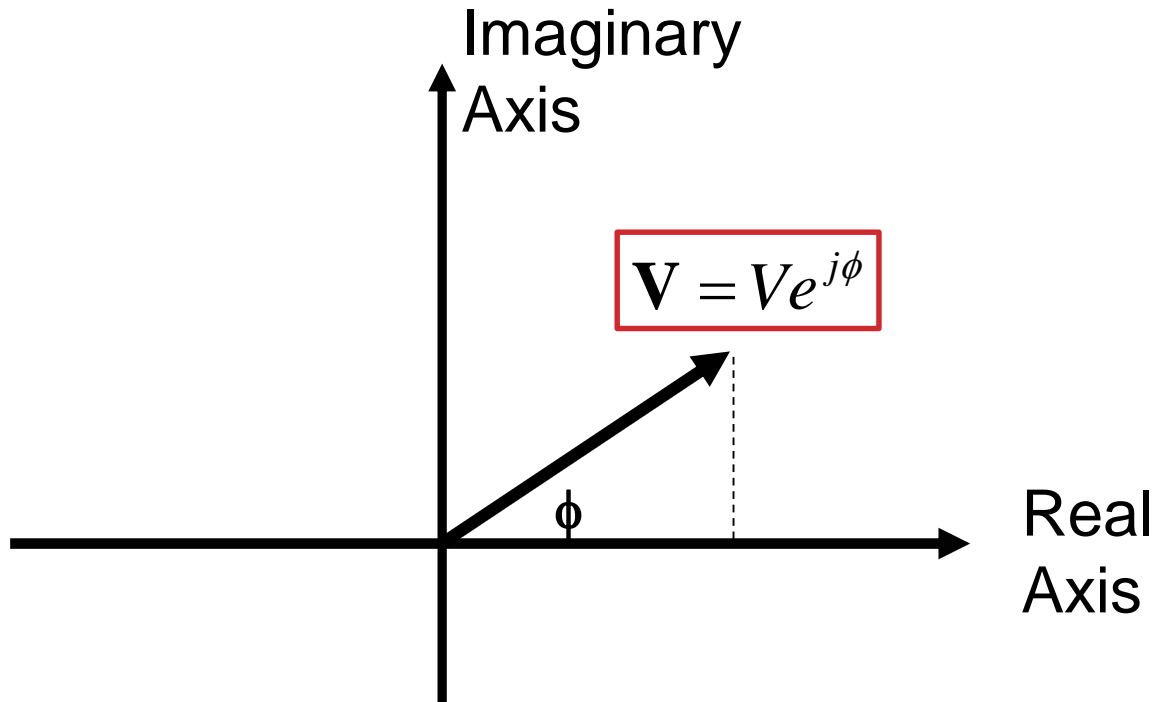
$$\sin \phi = \operatorname{Im}(e^{j\phi})$$



## Phasor:

$$v(t) = V \cos(\omega t + \phi) = \operatorname{Re}\{V e^{j\phi} e^{j\omega t}\} = \operatorname{Re}(\mathbf{V} e^{j\omega t}) \quad \boxed{\mathbf{V} = V e^{j\phi}}$$

Complex representation of the magnitude and phase of a **sinusoid**





# Phasor : Complex Numbers

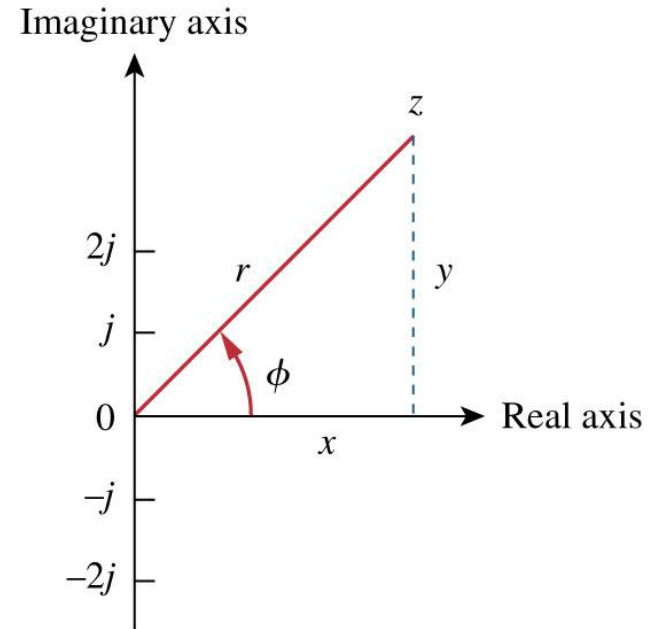
- A powerful method for representing sinusoids is the phasor, a complex expression as well.
- A complex number  $z$  can be represented in *polar* or *exponential* form as:

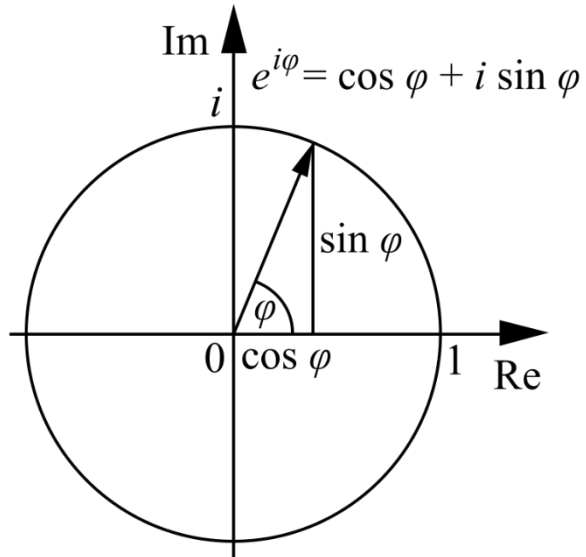
$$z = r \angle \phi = r e^{j\phi}$$

- It can also be written in *rectangular form* as:

$$z = x + jy \quad \begin{aligned} \operatorname{Re}(z) &= x \\ \operatorname{Im}(z) &= y \end{aligned}$$

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$$1 = 1e^{j0} = 1\angle 0^\circ$$

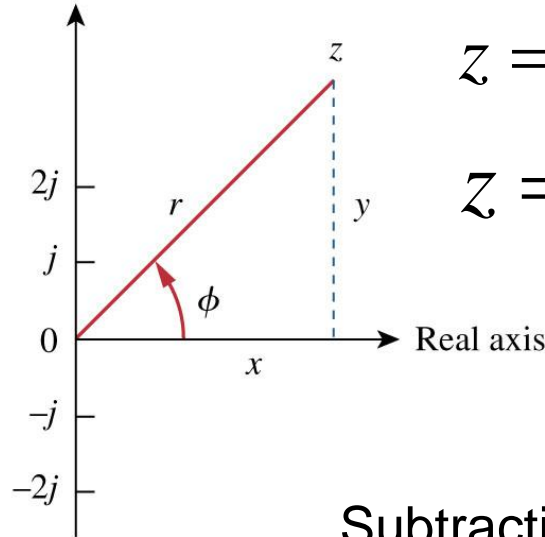
$$j = 1e^{j\frac{\pi}{2}} = 1\angle 90^\circ$$



# Arithmetic With Complex Numbers

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Imaginary axis



$$z = r \angle \phi = r e^{j\phi}$$

$$z = x + jy$$

Addition

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication

$$z_1 z_2 = r_1 r_2 \angle (\phi_1 + \phi_2)$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle (\phi_1 - \phi_2)$$

Reciprocal

$$\frac{1}{z} = \frac{1}{r} \angle (-\phi)$$

Square Root

$$z^{1/2} = \pm |z|^{1/2} e^{j\theta/2}$$

Complex Conjugate

$$z^* = x - jy = r \angle -\phi = r e^{-j\phi}$$



# Relations for Complex Numbers

<b>Euler's Identity:</b> $e^{j\theta} = \cos \theta + j \sin \theta$	
$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$
$\mathbf{z} = x + jy =  \mathbf{z} e^{j\theta}$	$\mathbf{z}^* = x - jy =  \mathbf{z} e^{-j\theta}$
$x = \Re(\mathbf{z}) =  \mathbf{z}  \cos \theta$	$ \mathbf{z}  = \sqrt[4]{\mathbf{z}\mathbf{z}^*} = \sqrt{x^2 + y^2}$
$y = \Im(\mathbf{z}) =  \mathbf{z}  \sin \theta$	$\theta = \tan^{-1}(y/x)$
$\mathbf{z}^n =  \mathbf{z} ^n e^{jn\theta}$	$\mathbf{z}^{1/2} = \pm  \mathbf{z} ^{1/2} e^{j\theta/2}$
$\mathbf{z}_1 = x_1 + jy_1$	$\mathbf{z}_2 = x_2 + jy_2$
$\mathbf{z}_1 = \mathbf{z}_2$ iff $x_1 = x_2$ and $y_1 = y_2$	$\mathbf{z}_1 + \mathbf{z}_2 = (x_1 + x_2) + j(y_1 + y_2)$
$\mathbf{z}_1 \mathbf{z}_2 =  \mathbf{z}_1   \mathbf{z}_2  e^{j(\theta_1 + \theta_2)}$	$\frac{\mathbf{z}_1}{\mathbf{z}_2} = \frac{ \mathbf{z}_1 }{ \mathbf{z}_2 } e^{j(\theta_1 - \theta_2)}$
$-1 = e^{j\pi} = e^{-j\pi} = 1 \angle \pm 180^\circ$	
$j = e^{j\pi/2} = 1 \angle 90^\circ$	$-j = e^{-j\pi/2} = 1 \angle -90^\circ$
$\sqrt{j} = \pm e^{j\pi/4} = \pm \frac{(1 + j)}{\sqrt{2}}$	$\sqrt{-j} = \pm e^{-j\pi/4} = \pm \frac{(1 - j)}{\sqrt{2}}$



## Example

- Evaluate these complex numbers

$$(a) (40 \angle 50^\circ + 20 \angle -30^\circ)^{1/2}$$

$$(b) \frac{10 \angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$$



## Exercise

- Evaluate the following complex numbers

(a)  $[(5 + j2)(-1 + j4) - 5\angle 60^\circ]^*$

(b)  $\frac{10 + j5 + 3\angle 40^\circ}{-3 + j4} + 10\angle 30^\circ + j5$





## Example

- Transform these sinusoids to phasors

(a)  $i = 6 \cos(50t - 40^\circ) \text{ A}$

(b)  $v = -4 \sin(30t + 50^\circ) \text{ V}$



## Example

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20^\circ} \text{ V}$$

$$\mathbf{V} = -25 \angle 40^\circ \text{ V}$$

$$\mathbf{I} = j(12 - j5) \text{ A}$$





# Time Domain - Phasor Domain Transformation

$x(t)$		$\mathbf{X}$
$A \cos \omega t$	$\longleftrightarrow$	$A$
$A \cos(\omega t + \phi)$	$\longleftrightarrow$	$Ae^{j\phi}$
$-A \cos(\omega t + \phi)$	$\longleftrightarrow$	$Ae^{j(\phi \pm \pi)}$
$A \sin \omega t$	$\longleftrightarrow$	$Ae^{-j\pi/2} = -jA$
$A \sin(\omega t + \phi)$	$\longleftrightarrow$	$Ae^{j(\phi - \pi/2)}$
$-A \sin(\omega t + \phi)$	$\longleftrightarrow$	$Ae^{j(\phi + \pi/2)}$
$\frac{d}{dt}(x(t))$	$\longleftrightarrow$	$j\omega \mathbf{X}$
$\frac{d}{dt}[A \cos(\omega t + \phi)]$	$\longleftrightarrow$	$j\omega Ae^{j\phi}$
$\int x(t) dt$	$\longleftrightarrow$	$\frac{1}{j\omega} \mathbf{X}$
$\int A \cos(\omega t + \phi) dt$	$\longleftrightarrow$	$\frac{1}{j\omega} Ae^{j\phi}$

It is much easier to deal with exponentials in the phasor domain than sinusoidal relations in the time domain.

- You just need to track magnitude/phase, knowing that everything is at frequency  $\omega$ .

# Phasor Relationships for Resistors

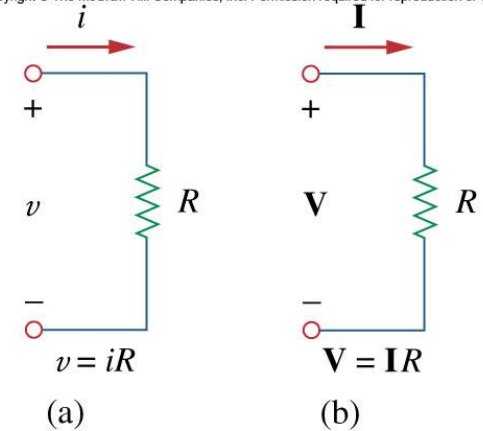
- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

$$i = I_m \cos(\omega t + \phi) \quad \mathbf{I} = I_m \angle \phi$$

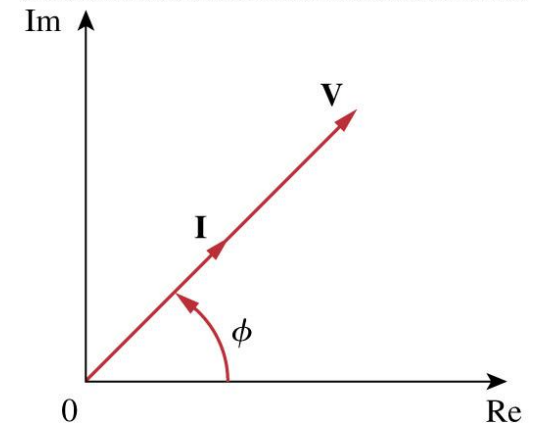
$$v = RI_m \cos(\omega t + \phi) \quad \mathbf{V} = RI_m \angle \phi$$

$$\mathbf{V} = R\mathbf{I}$$

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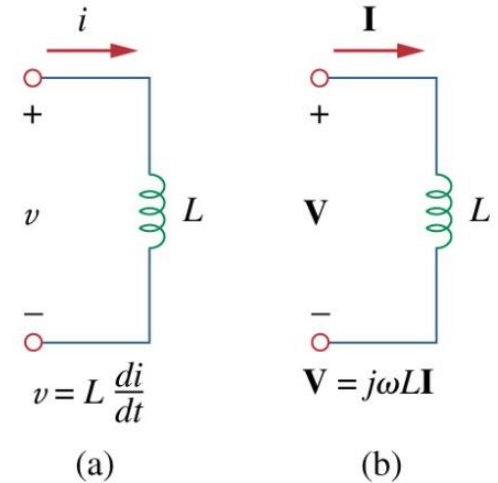


# Phasor Relationships for Inductors

$$i = I_m \cos(\omega t + \phi) \quad \mathbf{I} = I_m \angle \phi$$

$$v = L \frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \\ = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

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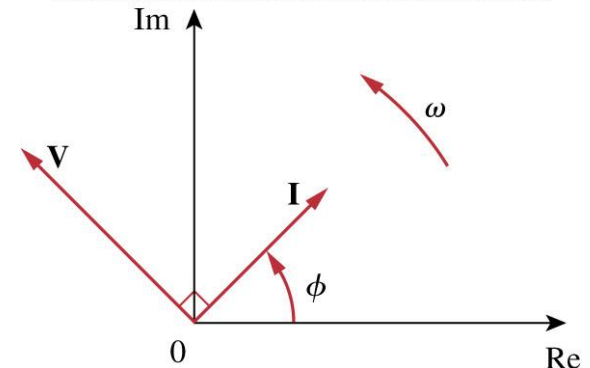


$$\mathbf{V} = \omega L I_m \angle \phi + 90^\circ = \omega L I_m e^{j(\phi + 90^\circ)} = \omega L e^{j90^\circ} \cdot I_m e^{j\phi} = j\omega L \cdot \mathbf{I}$$

$$\mathbf{V} = j\omega L \mathbf{I}$$

- The voltage leads the current by  $90^\circ$  (phase shift =  $90^\circ$ )

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# Phasor Relationships for Capacitors

$$v = V_m \cos(\omega t + \phi) \quad \mathbf{V} = V_m \angle \phi$$

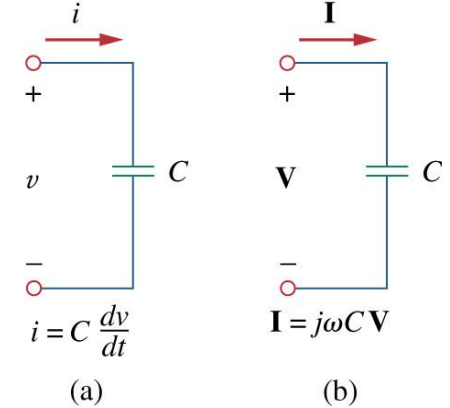
$$i = C \frac{dv}{dt} = -\omega C V_m \sin(\omega t + \phi) \\ = \omega C V_m \cos(\omega t + \phi + 90^\circ)$$

$$\mathbf{I} = \omega C V_m \angle \phi + 90^\circ = \dots = j\omega C \cdot \mathbf{V}$$

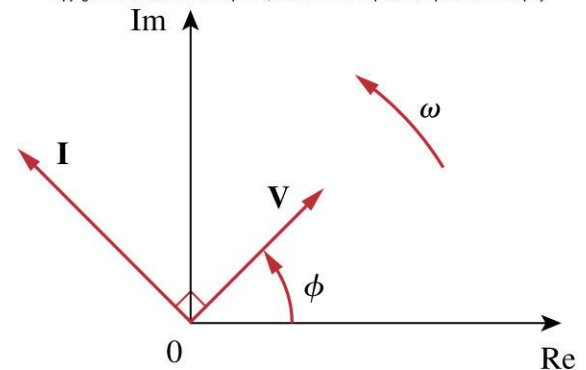
$$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

- The voltage lags the current by  $90^\circ$ .

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# Impedance

- The voltage-current relations for R, L and C elements are

$$\begin{array}{ccc} \mathbf{V} = R\mathbf{I} & \mathbf{V} = j\omega L\mathbf{I} & \mathbf{V} = \frac{\mathbf{I}}{j\omega C} \\ & \downarrow & \\ \frac{\mathbf{V}}{\mathbf{I}} = R & \frac{\mathbf{V}}{\mathbf{I}} = j\omega L & \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C} \end{array}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\mathbf{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- Z** is called **impedance**, measured in ohms.
  - Impedance is not a phasor! But it is (often) a complex number.
  - Impedance depends on the frequency  $\omega$ .



# Admittance





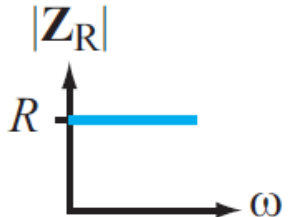
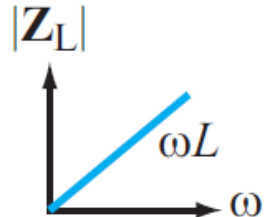
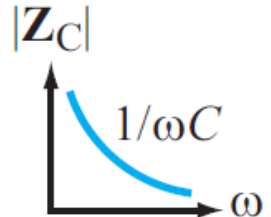
Admittance is simply the inverse of impedance, unit: Simens.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

Element	Impedance	Admittance
$R$	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
$L$	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
$C$	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$



# Summary of $R, L, C$

Property	$R$	$L$	$C$
$v-i$	$v = Ri$	$v = L \frac{di}{dt}$	$i = C \frac{dv}{dt}$
<b>V-I</b>	<b><math>\mathbf{V} = R\mathbf{I}</math></b>	<b><math>\mathbf{V} = j\omega L\mathbf{I}</math></b>	<b><math>\mathbf{V} = \frac{\mathbf{I}}{j\omega C}</math></b>
<b>Z</b>	$R$	$j\omega L$	$\frac{1}{j\omega C}$
dc equivalent	$R$	 Short circuit	 Open circuit
High-frequency equivalent	$R$	 Open circuit	 Short circuit
Frequency response			



## Recall-- Sinusoidal Response

$$v_s = V_m \cos(\omega t + \phi), i(0^-) = 0.$$

Find  $i(t)$ ,  $t \geq 0$ .

$$L \frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

