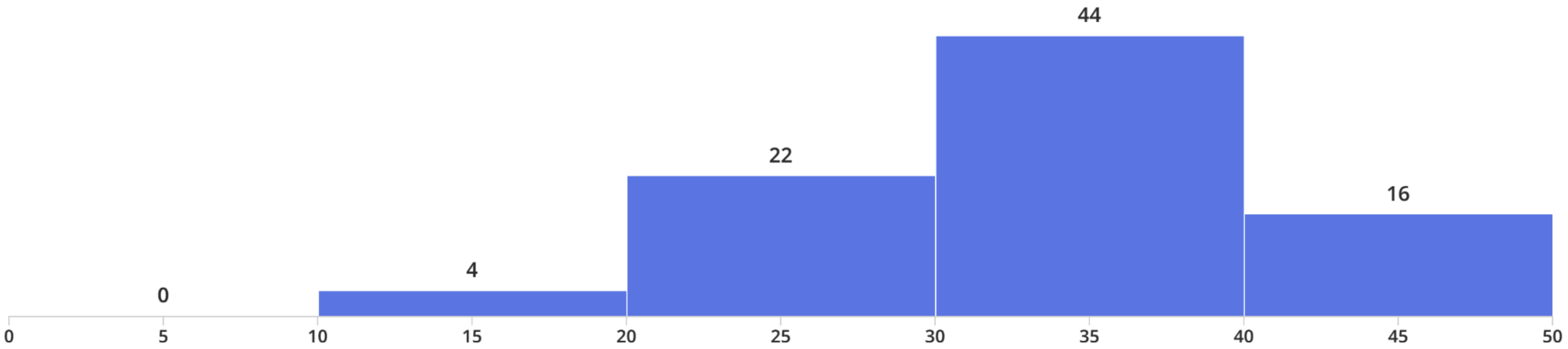
Midterm



Minimum	Median	Maximum	Mean	Std Dev
15.0	34.0	46.0	32.93	6.83

- Request regrading on Gradescope by next Thu

Project

- Group registration
  - Register your group members by Nov. 22
    - BB → Project → Project group registration
    - Each group only needs to register once
  - No more than 5 people in a group. Talk to me if you wish to form a larger group
- Proposal presentation
  - Dec. 4, 6
  - 5-10min per group (depending on the number of groups) including QA

Probabilistic Reasoning over Time

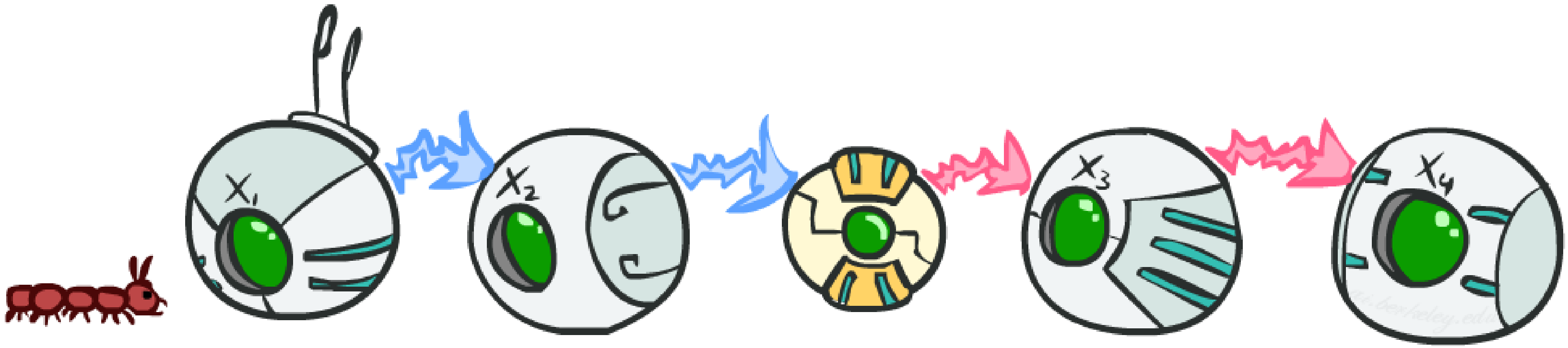
AIMA Chapter 15

[Adapted from slides by Dan Klein and Pieter Abbeel at UC Berkeley]

Uncertainty and Time

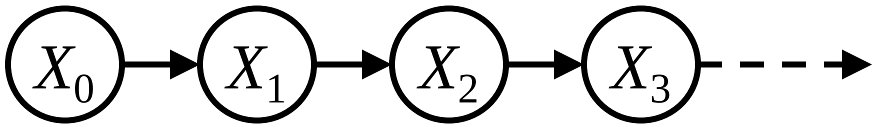
- Often, we want to reason about a ***sequence*** of observations
  - Speech recognition
  - Robot localization
  - Medical monitoring
  - User attention
- Need to introduce time into our models

Markov Models



Markov Models (aka Markov chain/process)

- Assume discrete variables that share the same finite domain
  - Values in the domain is called the ***states***

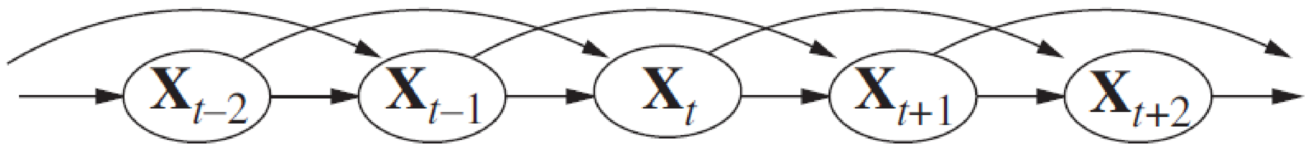


$P(X_0)$ $P(X_t \mid X_{t-1})$

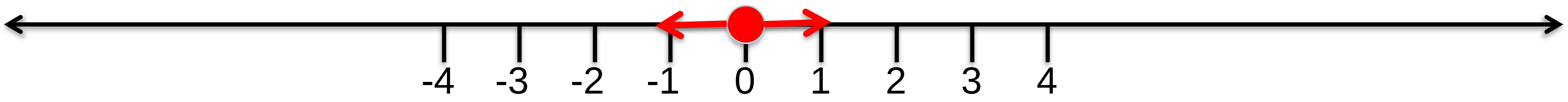
- The ***transition model*** $P(X_t \mid X_{t-1})$ specifies how the state evolves over time
- ***Stationarity*** assumption: same transition probabilities *at all time steps*
- Joint distribution $P(X_0,\ldots, X_T) = P(X_0) \prod_t P(X_t \mid X_{t-1})$

Markov Assumption: Conditional Independence

- ***Markov*** assumption: X_{t+1}, … are independent of $X_0,\ldots, X_{t-1}$ given X_t
  - Past and future independent given the present
  - Each time step only depends on the previous
- This is a ***first-order*** Markov model
- A *k*th-order model allows dependencies on *k* earlier steps



Example: Random walk in one dimension



- State: location on the unbounded integer line
- Initial probability: starts at 0
- Transition model: $P(X_t = k \mid X_{t-1} = k \pm 1) = 0.5$
- Applications: particle motion in crystals, stock prices, gambling, genetics, etc.

Example: n-gram models

- State: word at position t in text (can also build letter n-grams)
- Transition model
  - Unigram (zero-order): $P(Word_t = i)$
    - "logical are as are confusion a may right tries agent goal the was . . ."
  - Bigram (first-order): $P(Word_t = i \mid Word_{t\text{-}1} = j)$
    - "systems are very similar computational approach would be represented . . ."
  - Trigram (second-order): $P(Word_t = i \mid Word_{t\text{-}1} = j, Word_{t\text{-}2} = k)$
    - "planning and scheduling are integrated the success of naive bayes model is . . ."
- Applications: text classification, spam detection, author identification, language classification, speech recognition

Example: Weather

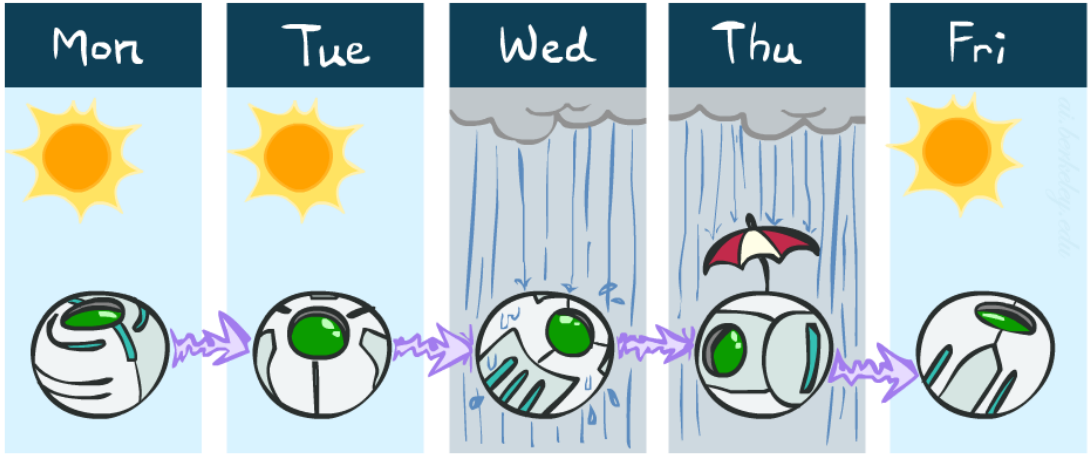
- States {rain, sun}

- Initial distribution $P(X_0)$

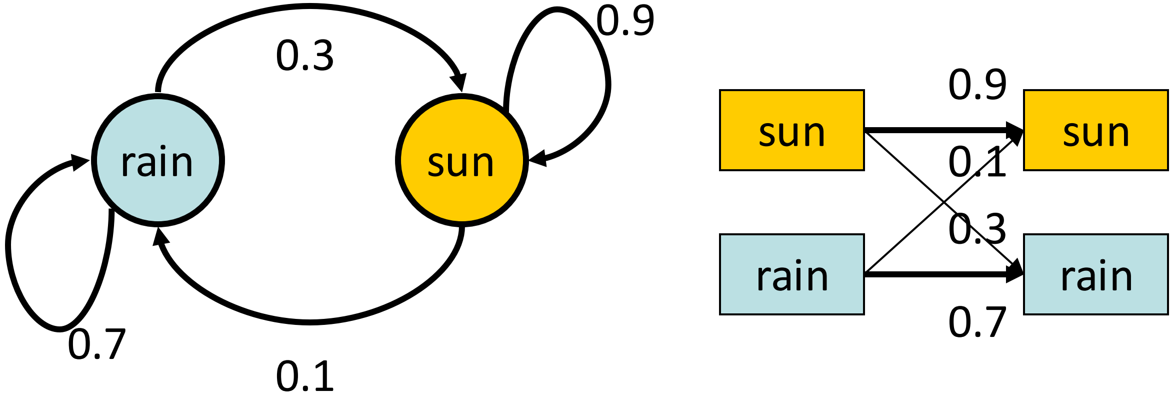
$P(X_0)$	
sun	rain
0.5	0.5

- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



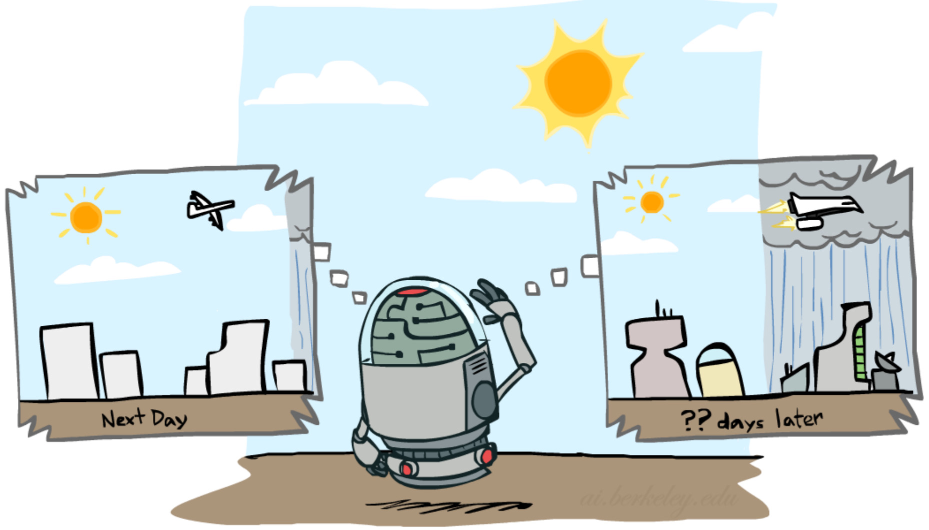
Two new ways of representing the same CPT



Weather prediction

- Time 0: <0.5,0.5>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

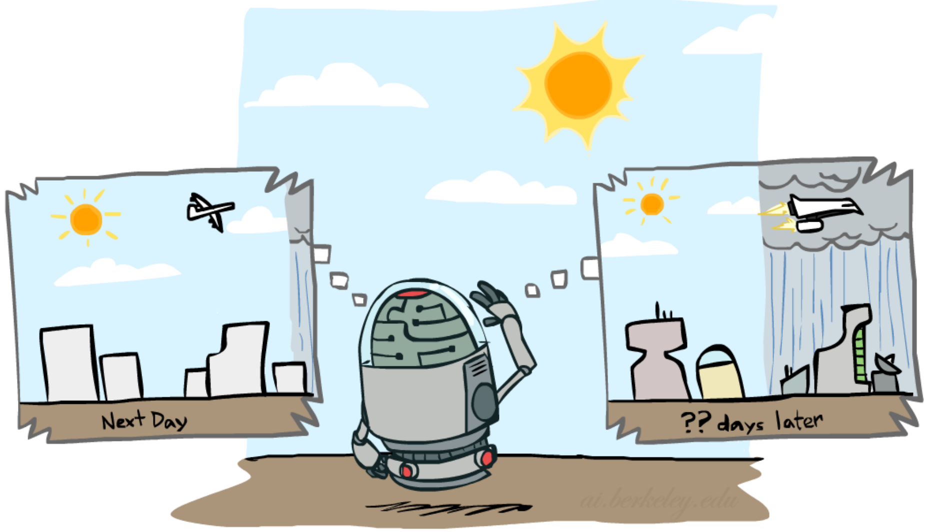


- What is the weather like at time 1?
  - $P(X_1) = \sum_{x_0} P(X_1, X_0 = x_0)$
  - $= \sum_{x_0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$
  - $= 0.5<0.9,0.1> + 0.5<0.3,0.7> = <0.6,0.4>$

Weather prediction, contd.

- Time 1: <0.6,0.4>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

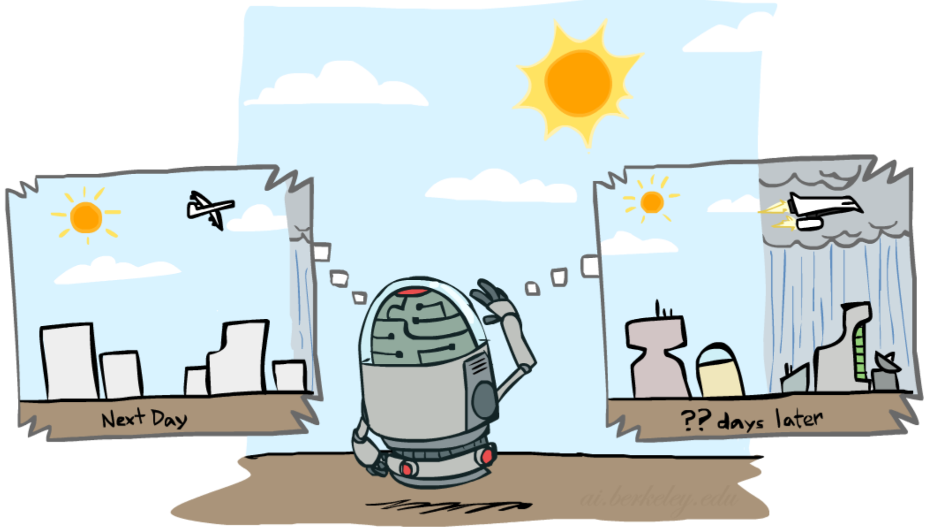


- What is the weather like at time 2?
  - $P(X_2) = \sum_{x_1} P(X_2, X_1{=}x_1)$
  - $= \sum_{x_1} P(X_1{=}x_1)\, P(X_2 \mid X_1{=}x_1)$
  - $= 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$

Weather prediction, contd.

- Time 2: <0.66,0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 3?
  - $P(X_3) = \sum_{x_2} P(X_3, X_2 = x_2)$
  - $\quad = \sum_{x_2} P(X_2 = x_2)\, P(X_3 \mid X_2 = x_2)$
  - $\quad = 0.66\langle 0.9, 0.1\rangle + 0.34\langle 0.3, 0.7\rangle = \langle 0.696, 0.304\rangle$

Forward algorithm (simple form)

- What is the state at time *t* (given an initial distribution $P(X_0)$)?
  - $P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$
  - $= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1})\, P(X_t \mid X_{t-1}=x_{t-1})$

Probability from previous iteration

Transition model

- Iterate this update starting at $t=0$

And the same thing in linear algebra

- What is the weather like at time 2?
  - $P(X_2) = 0.6\langle 0.9,0.1\rangle + 0.4\langle 0.3,0.7\rangle = \langle 0.66,0.34\rangle$
- In matrix-vector form:
  - $P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- I.e., multiply by T^{T}, transpose of transition matrix

Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix} \quad \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} \quad \begin{pmatrix} 0.84 \\ 0.16 \end{pmatrix} \quad \begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

$$\mathrm{P}(X_0) \quad \mathrm{P}(X_1) \quad \mathrm{P}(X_2) \quad \mathrm{P}(X_3) \quad \mathrm{P}(X_\infty)$$

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

- From initial observation of rain

$$\begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix} \quad \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \quad \begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix} \quad \begin{pmatrix} 0.588 \\ 0.412 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

$$\mathrm{P}(X_0) \quad \mathrm{P}(X_1) \quad \mathrm{P}(X_2) \quad \mathrm{P}(X_3) \quad \mathrm{P}(X_\infty)$$

- From yet another initial distribution $P(X_0)$:

$$\begin{pmatrix} p \\ 1-p \end{pmatrix} \quad \cdots \quad \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

$$\mathrm{P}(X_0) \qquad \mathrm{P}(X_\infty)$$

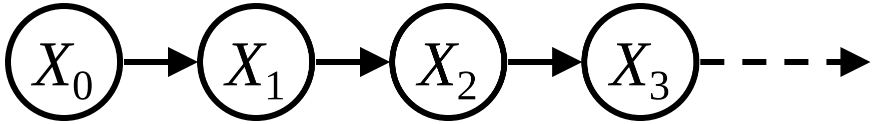
Stationary Distributions

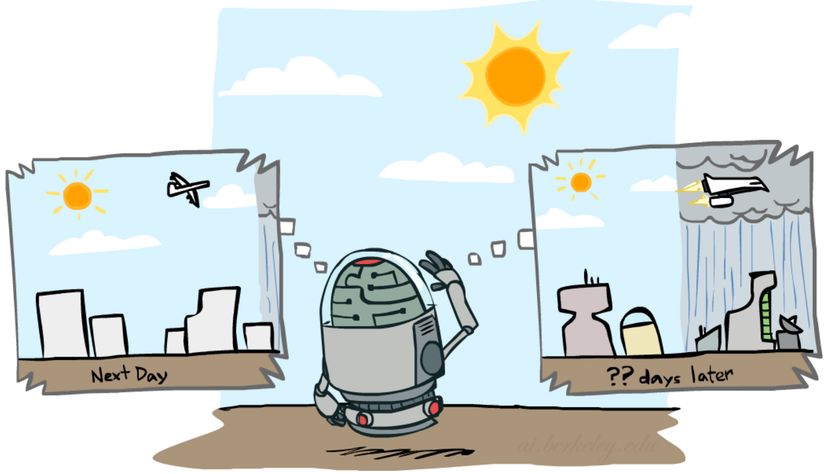
- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution
- Stationary distribution:
  - The distribution we end up with is called the stationary distribution P_∞ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Example: Stationary Distributions

- Computing the stationary distribution





$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

Also: $P_\infty(sun) + P_\infty(rain) = 1$

$$\Rightarrow \quad P_\infty(sun) = 3/4 \qquad P_\infty(rain) = 1/4$$

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Stationary Distributions

- The limiting distribution is called the ***stationary distribution*** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^T P_\infty$
- Solving for P_∞ in the example:

$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

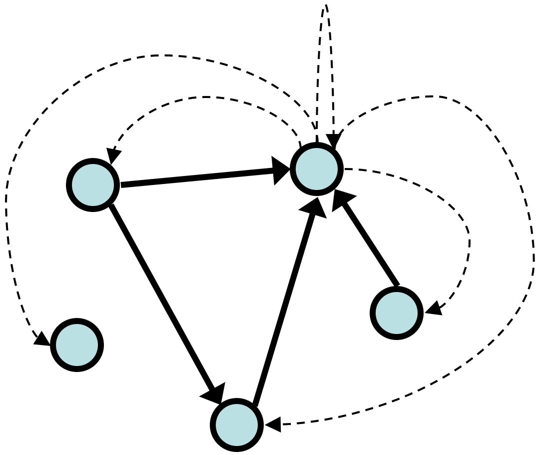
$0.9p + 0.3(1-p) = p$

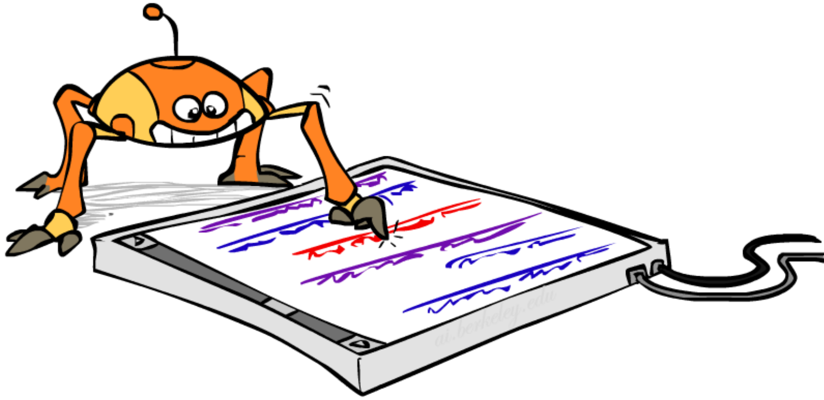
$p = 0.75$

Stationary distribution is <0.75,0.25> ***regardless of starting distribution***

Application of Stationary Distribution: Web Link Analysis

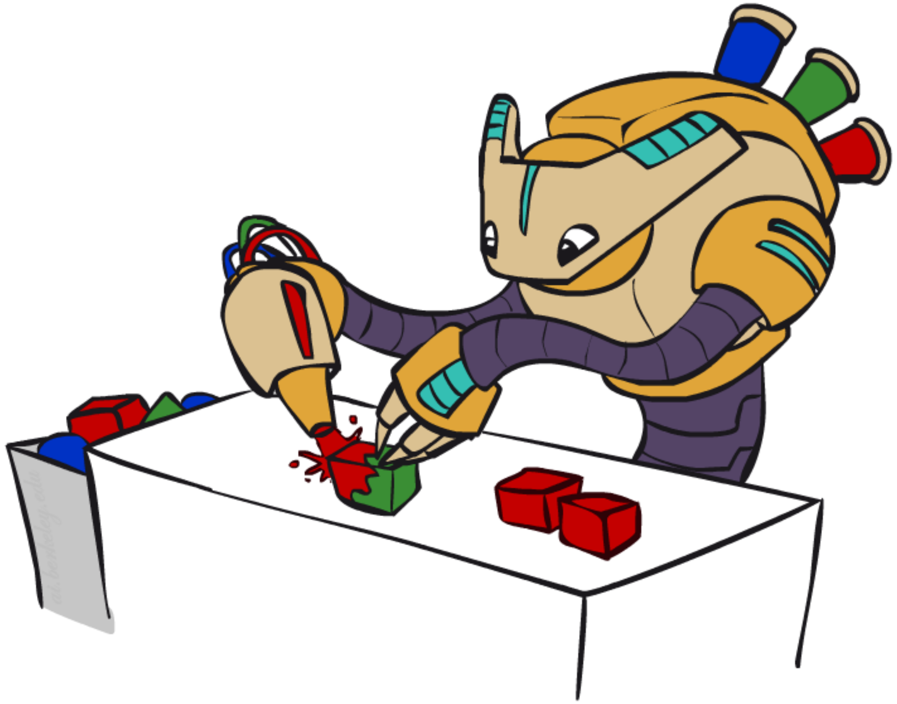
- Web browsing
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page
    - With prob. 1-c, follow a random outlink
- Stationary distribution: PageRank
  - Will spend more time on highly reachable pages
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank
  - Now: use link analysis along with many other factors (rank actually getting less important)





Application of Stationary Distributions: Gibbs Sampling

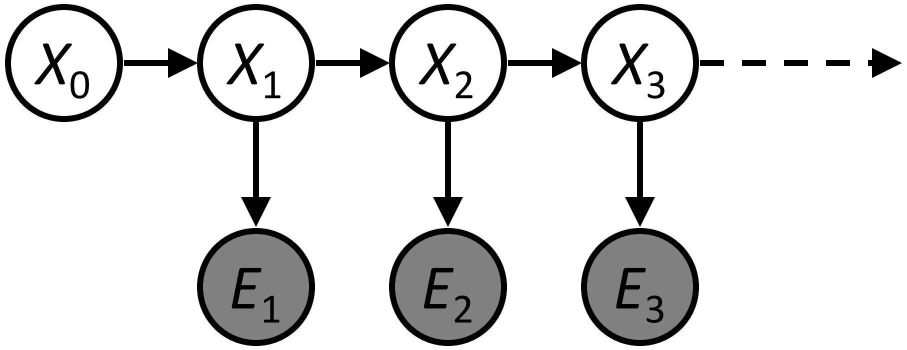
- Each joint instantiation over all hidden and query variables is a state: $\{X_1, \ldots, X_n\} = H \cup Q$
- Transitions:
  - Pick a variable and resample its value conditioned on its Markov blanket
- Stationary distribution:
  - Conditional distribution $P(X_1, X_2, \ldots, X_n | e_1, \ldots, e_m)$
  - When running Gibbs sampling long enough, we get a sample from the desired distribution



Hidden Markov Models

Hidden Markov Models

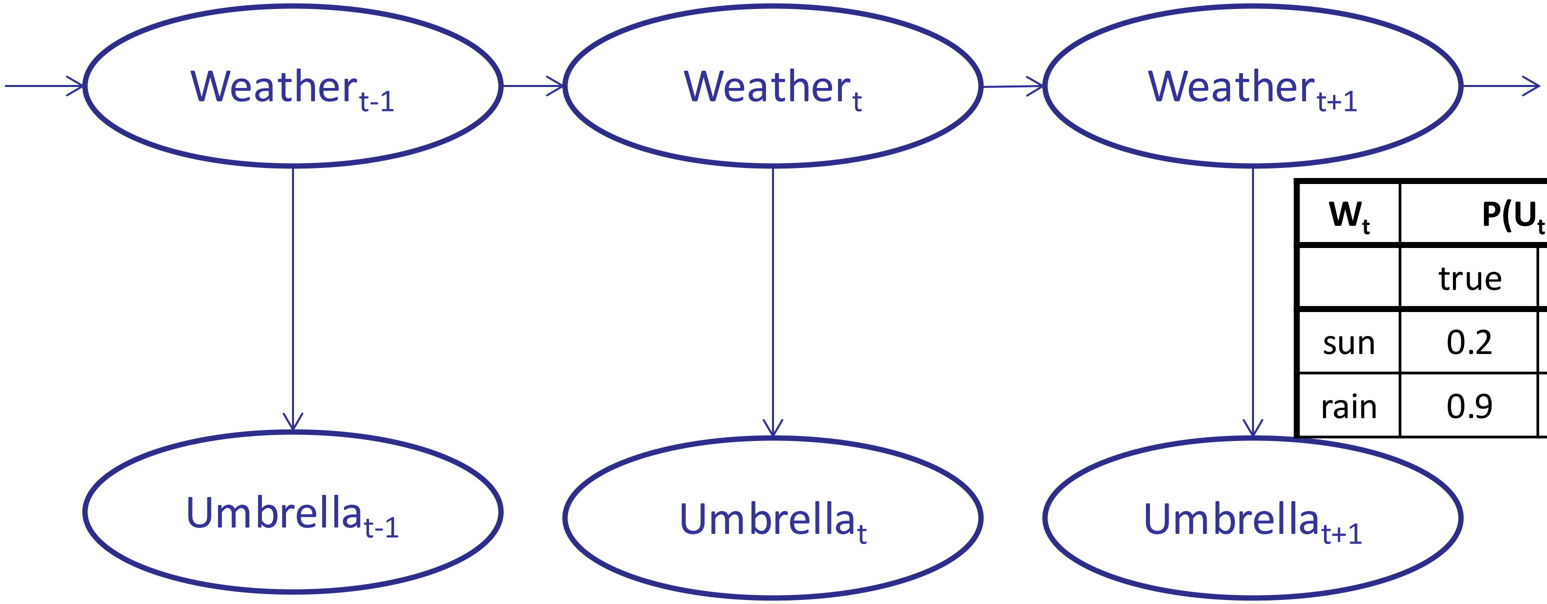
- Usually the true state is not observed directly
  - E.g., you stay indoor and cannot see the weather, but you can see if people come in with umbrella or not.
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe evidence E at each time step



Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t | X_{t-1})$
  - Emission model: $P(E_t | X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



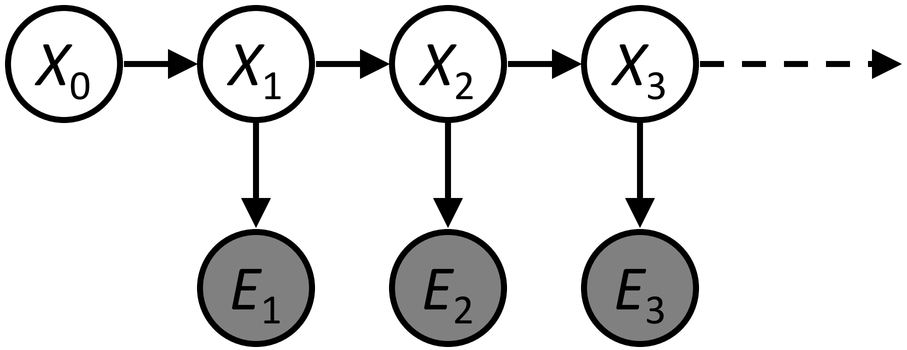
W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

HMM as probability model

- Joint distribution for Markov model: $P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$
- Joint distribution for hidden Markov model:

  $P(X_0,X_1, E_1,\ldots, X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1}) P(E_t \mid X_t)$

- Independence in HMM
  - Future states are independent of the past given the present
  - Current evidence is independent of everything else given the current state



Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Inference tasks

- Useful notation: $X_{a:b} = X_a, X_{a+1}, \ldots, X_b$
- ***Filtering***: $P(X_t | e_{1:t})$
  - ***belief state*** — posterior distribution over the most recent state given all evidence
  - Ex: robot localization
- ***Prediction***: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - posterior distribution over a future state given all evidence
- ***Smoothing***: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - posterior distribution over a past state given all evidence
- ***Most likely explanation***: $\arg\max_{x_{0:t}} P(x_{0:t} \mid e_{1:t})$
  - Ex: speech recognition, decoding with a noisy channel

Filtering

- Filtering: infer current state given all evidence
- Aim: a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

Apply Bayes' rule

- $P(X_{t+1} | e_{1:t+1}) = P(\underline{X_{t+1}} | e_{1:t}, \underline{e_{t+1}})$
- $= \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$

$\alpha = 1 / P(e_{t+1} | e_{1:t})$

Filtering

- Filtering: infer current state given all evidence
- Aim: a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$
- $= \alpha\, P(e_{t+1} | \underline{X_{t+1}, e_{1:t}})\, P(X_{t+1} | e_{1:t})$
- $= \underbrace{\alpha}_{\text{Normalize}}\, \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}}\, \underbrace{P(X_{t+1} | e_{1:t})}_{\text{Predict}}$

Apply conditional independence

Filtering

- Filtering: infer current state given all evidence
- Aim: a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$
- $= \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1})\, \underline{P(X_{t+1} | e_{1:t})}$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | x_t, e_{1:t})$

Condition on X_t

Filtering

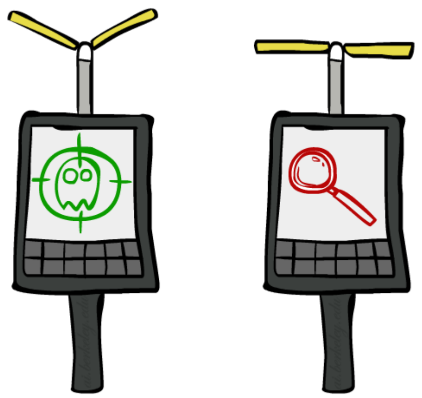
- Filtering: infer current state given all evidence
- Aim: a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

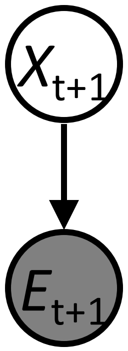
- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$
- $= \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | \underline{x_t, e_{1:t}})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | x_t)$

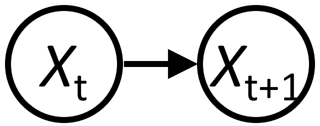
Apply conditional independence

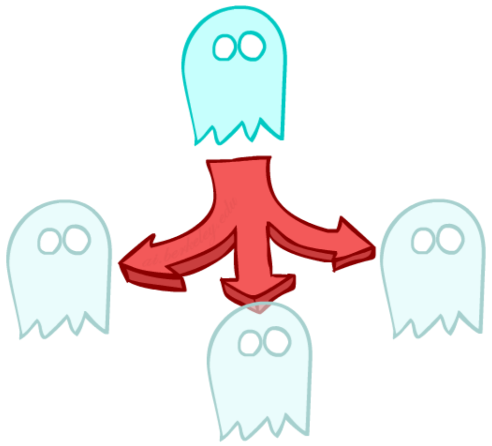
Filtering

- $P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) \, P(X_{t+1} | x_t)}_{\text{Predict}}$









Forward algorithm

- $P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \; \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \; \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) \, P(X_{t+1} | \, x_t)}_{\text{Predict}}$

- $\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}\,,\, e_{t+1})$
- We start with $\boldsymbol{f}_{1:0} = P(X_0)$ and then iterate
- Cost *per time step*: $O(|X|^2)$ where $|X|$ is the number of states

Example: Weather HMM

$$P(X_{t+1}|e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}}\ \underbrace{P(e_{t+1}|X_{t+1})}_{\text{Update}}\ \underbrace{\sum_{x_t} P(x_t \mid e_{1:t})\, P(X_{t+1} \mid x_t)}_{\text{Predict}}$$

f(sun) = 0.5
f(rain) = 0.5

predict

0.6
0.4

$P(s) = 0.5 \times 0.9 + 0.5 \times 0.3 = 0.6$

$P(r) = 0.5 \times 0.1 + 0.5 \times 0.7 = 0.4$

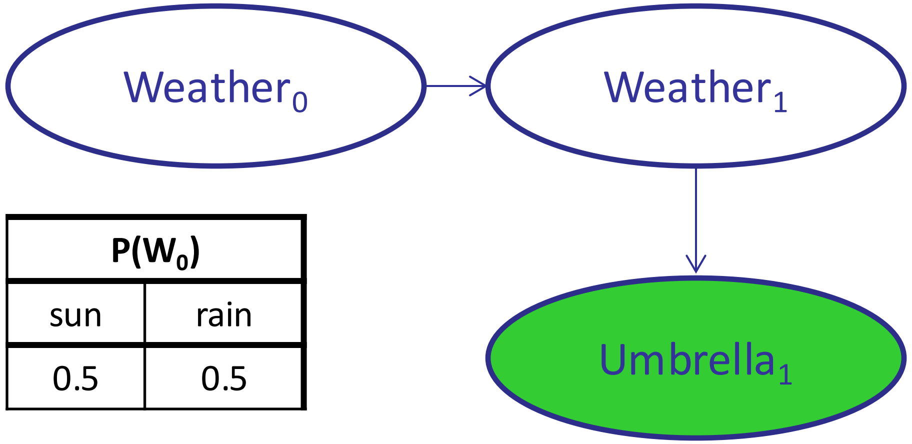
update & normalize

f(sun) = 0.25
f(rain) = 0.75

$P(s|u) \propto 0.6 \times 0.2 = 0.12$

$P(r|u) \propto 0.4 \times 0.9 = 0.36$

W_{t-1}	$P(W_t \| W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

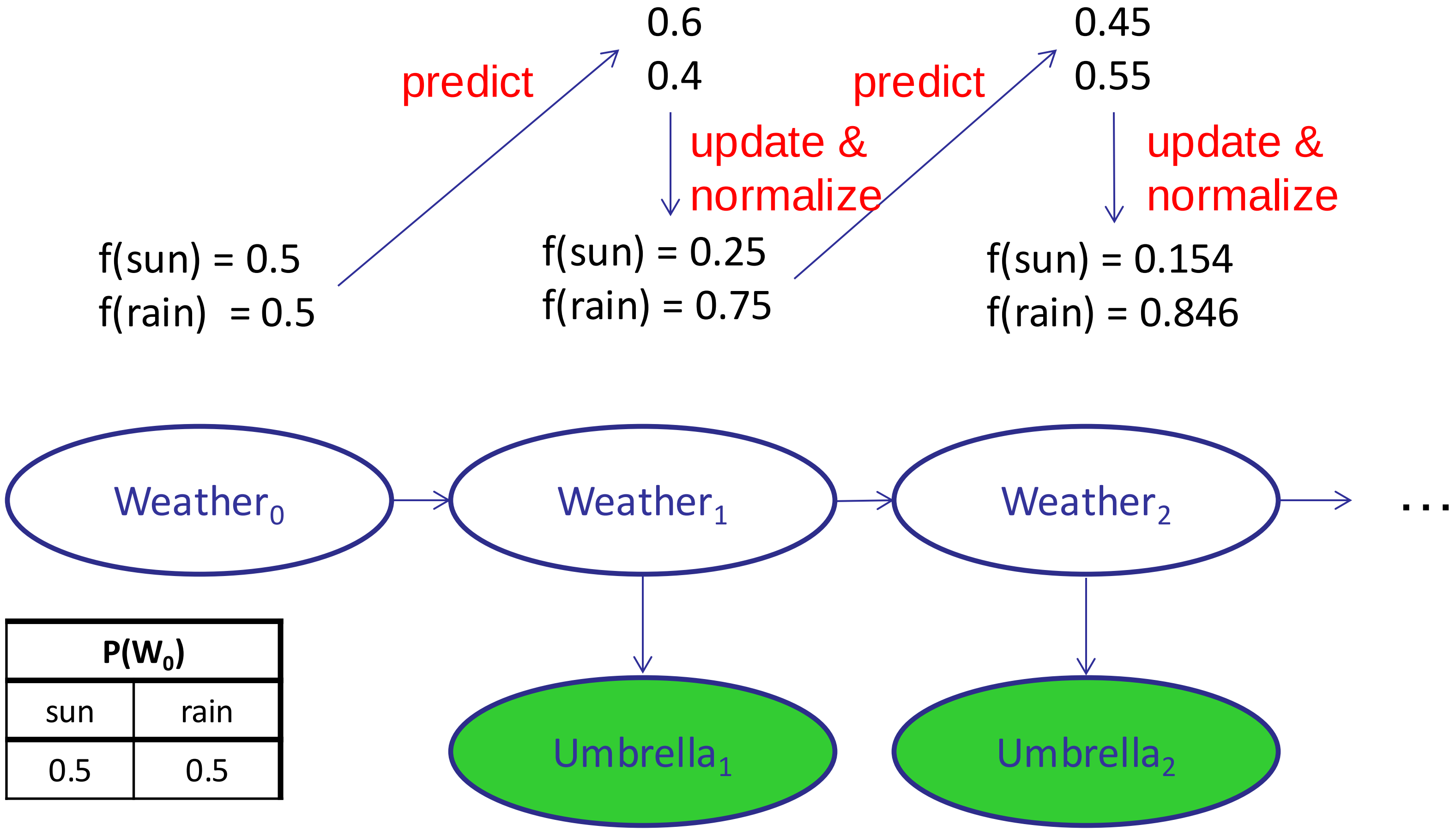


$P(W_0)$	
sun	rain
0.5	0.5

W_t	$P(U_t \| W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Weather HMM

$$P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) P(X_{t+1} | x_t)}_{\text{Predict}}$$



P(W_0)	
sun	rain
0.5	0.5

W_{t-1}	P(W_t | W_{t-1})	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	P(U_t | W_t)	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Forward algorithm

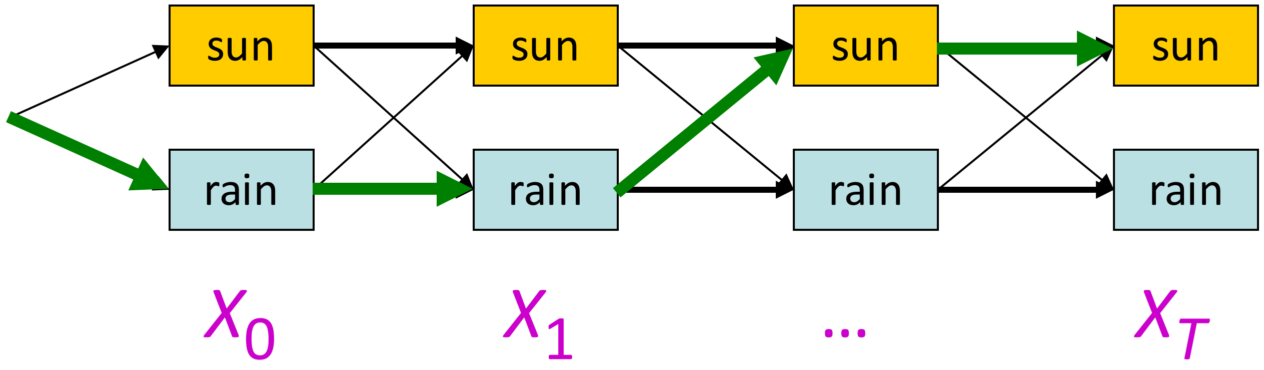
- $P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) P(X_{t+1} | x_t)}_{\text{Predict}}$

α is a constant. So if we only want to compute $P(x_t \mid e_{1:t})$, then we can skip normalization when computing $P(x_1 \mid e_1)$, $P(x_2 \mid e_{1:2})$, …, $P(x_{t-1} \mid e_{1:t-1})$

Q: How is the algorithm related to variable elimination?

Another view of the algorithm

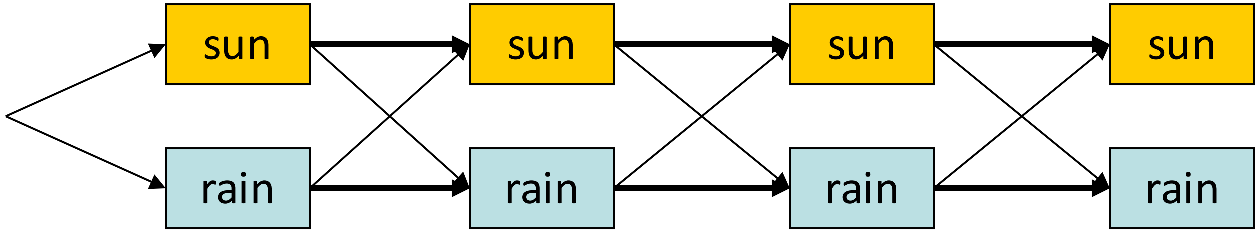
- ***State trellis***: graph of states and transitions over time



- Each arc represents some transition $x_{t-1} \rightarrow x_t$
- Each arc has weight $P(x_t \mid x_{t-1})\ P(e_t \mid x_t)$ (arcs to initial states have weight $P(x_0)$)
- Each path is a sequence of states
- The ***product*** of weights on a path is proportional to that state sequence's probability

$$P(x_0) \prod_t P(x_t \mid x_{t-1})\ P(e_t \mid x_t) = P(x_{0:t}, e_{1:t}) \propto P(x_{0:t} \mid e_{1:t})$$

Another view of the algorithm



X_0 X_1 … X_{t+1}

- Forward algorithm computes sum over all possible paths

  $P(x_{t+1} | e_{1:t+1}) = \sum_{x_{0:t}} P(x_{0:t+1} \mid e_{1:t+1})$
- It uses dynamic programming to sum over all paths
  - For each state at time t, keep track of the total probability of all paths to it

    $\boldsymbol{f}_{1:t+1} = \textsc{Forward}(\boldsymbol{f}_{1:t}, e_{t+1})$

    $= \alpha\, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1} | x_t)\, \boldsymbol{f}_{1:t}[x_t]$

    Aggregate the information from all variants of X_t

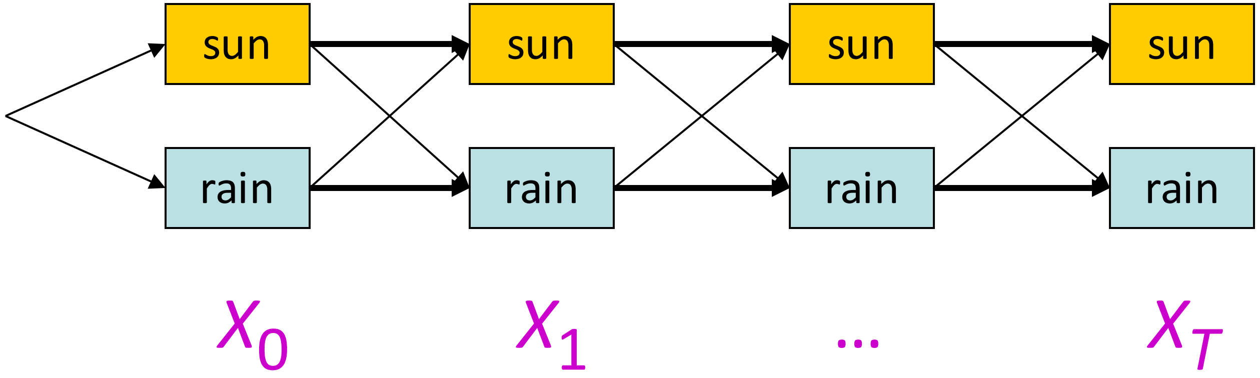
Most Likely Explanation

Inference tasks

- ***Filtering***: $P(X_t | e_{1:t})$
  - ***belief state***—input to the decision process of a rational agent
- ***Prediction***: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- ***Smoothing***: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- ***Most likely explanation***: $\arg\max_{x_{0:t}} P(x_{0:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Most likely explanation = most probable path

- ***State trellis***: graph of states and transitions over time



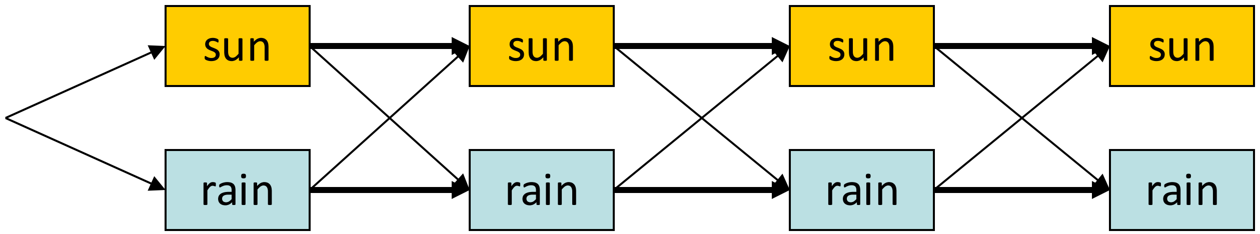
- The ***product*** of weights on a path is proportional to that state sequence's probability

  $P(x_0) \prod_t P(x_t \mid x_{t-1}) P(e_t \mid x_t) = P(x_{0:t}, e_{1:t}) \propto P(x_{0:t} \mid e_{1:t})$

- ***Viterbi algorithm*** computes best paths

  $\arg\max_{x_{0:t}} P(x_{0:t} \mid e_{1:t})$

Forward / Viterbi algorithms



X_0 X_1 … X_T

Viterbi Algorithm (max)

For each state at time t, keep track of the (unnormalized) ***maximum probability of any path*** to it:

$\boldsymbol{m}_{1:t}(x_t) = \underline{\max_{x_{1:t-1}}} \; \underline{P(x_{1:t} | e_{1:t})}$

$\boldsymbol{m}_{1:t+1} = \text{VITERBI}(\boldsymbol{m}_{1:t}, e_{t+1})$

$= P(e_{t+1}|X_{t+1}) \max_{x_t} P(X_{t+1}| x_t) \, \boldsymbol{m}_{1:t}[x_t]$

Forward Algorithm (sum)

For each state at time t, keep track of the ***total probability of all paths*** to it:

$\boldsymbol{f}_{1:t}(x_t) = P(x_t | e_{1:t})$

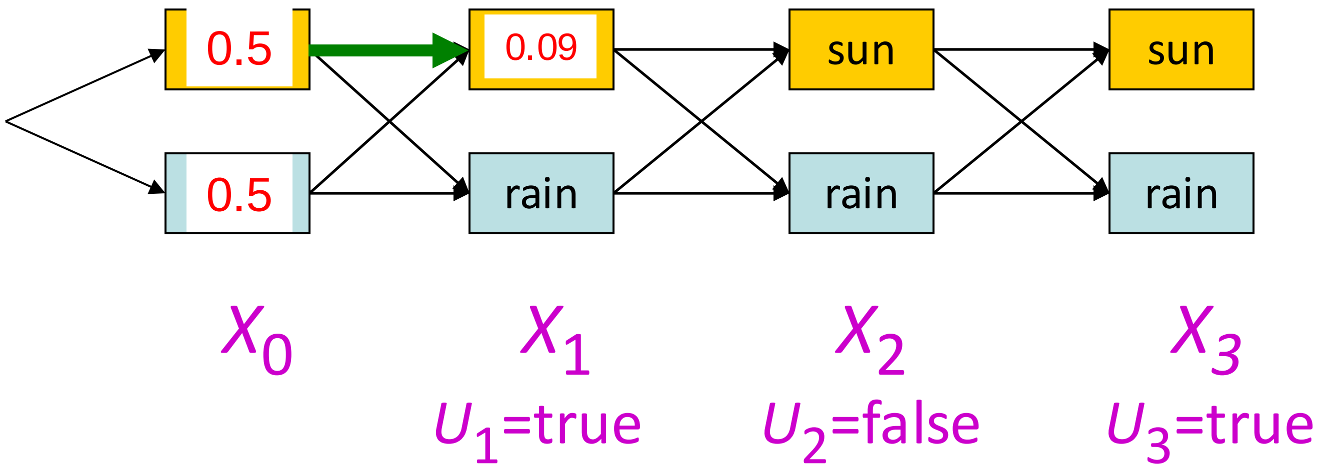
$= \sum_{x_{1:t-1}} P(x_{1:t} | e_{1:t})$

$\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}, e_{t+1})$

$= \alpha \, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}| x_t) \, \boldsymbol{f}_{1:t}[x_t]$

Viterbi algorithm contd.

P(W_0)	
sun	rain
0.5	0.5



W_{t-1}	P(W_t|W_{t-1})	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

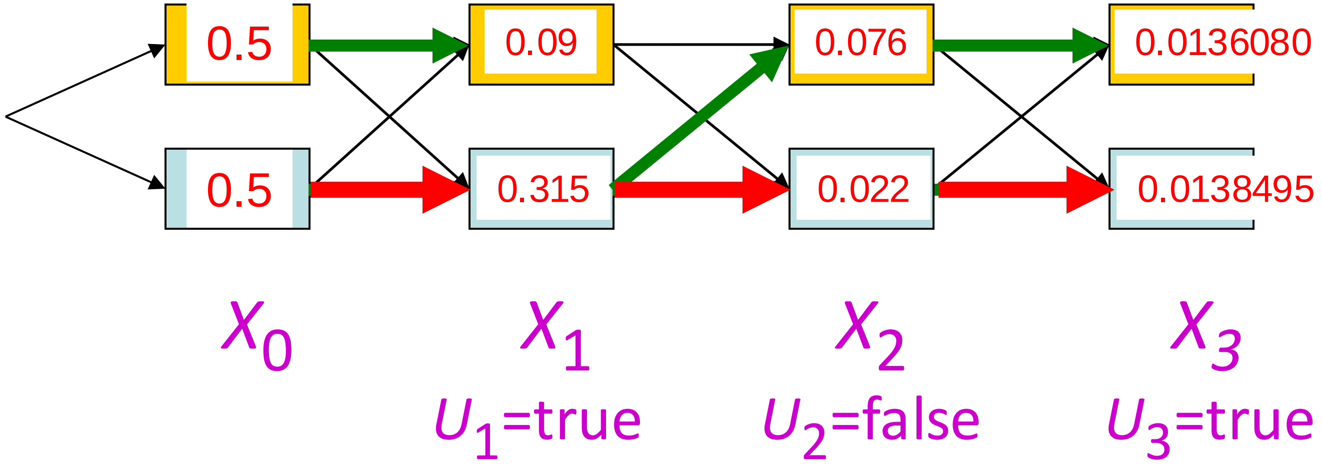
W_t	P(U_t|W_t)	
	true	false
sun	0.2	0.8
rain	0.9	0.1

- $\boldsymbol{m}_{1:t+1} = P(e_{t+1} | X_{t+1}) \max_{x_t} P(X_{t+1} | x_t)\, \boldsymbol{m}_{1:t}[x_t]$

$$m_{1:1}(\text{sun}) = 0.2 \times \max(\underline{0.9 \times 0.5}, 0.3 \times 0.5) = 0.09$$

Viterbi algorithm contd.

P(W_0)	
sun	rain
0.5	0.5



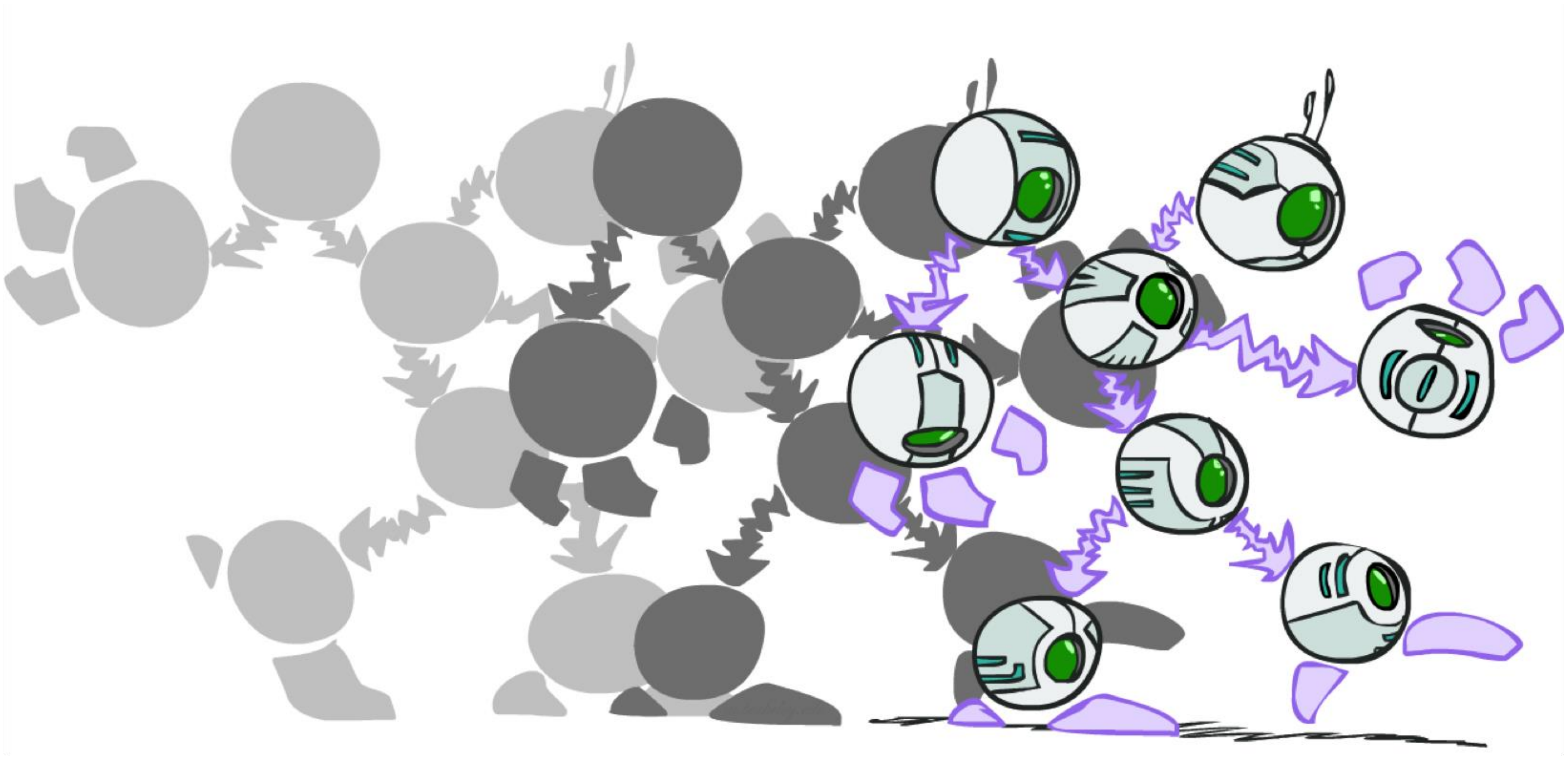
W_{t-1}	P(W_t|W_{t-1})	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	P(U_t|W_t)	
	true	false
sun	0.2	0.8
rain	0.9	0.1

- $\boldsymbol{m}_{1:t+1} = P(e_{t+1} | X_{t+1}) \max_{x_t} P(X_{t+1} | x_t) \, \boldsymbol{m}_{1:t} [x_t]$

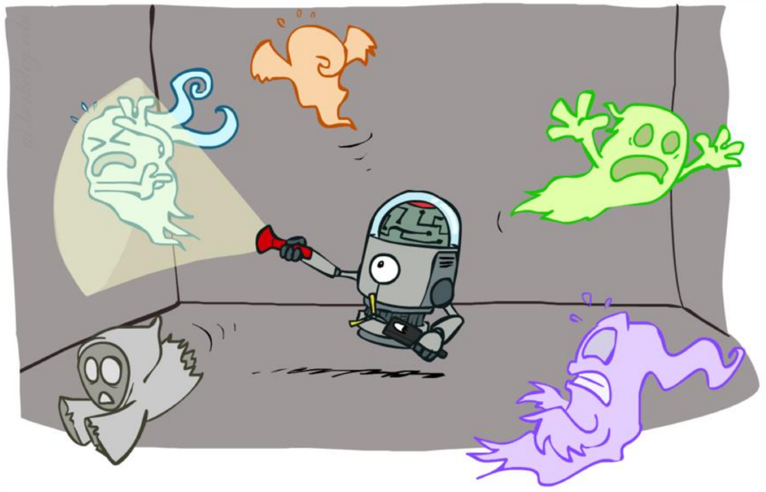
- Time complexity: **O(|X|² T)**
- Space complexity: **O(|X| T)**

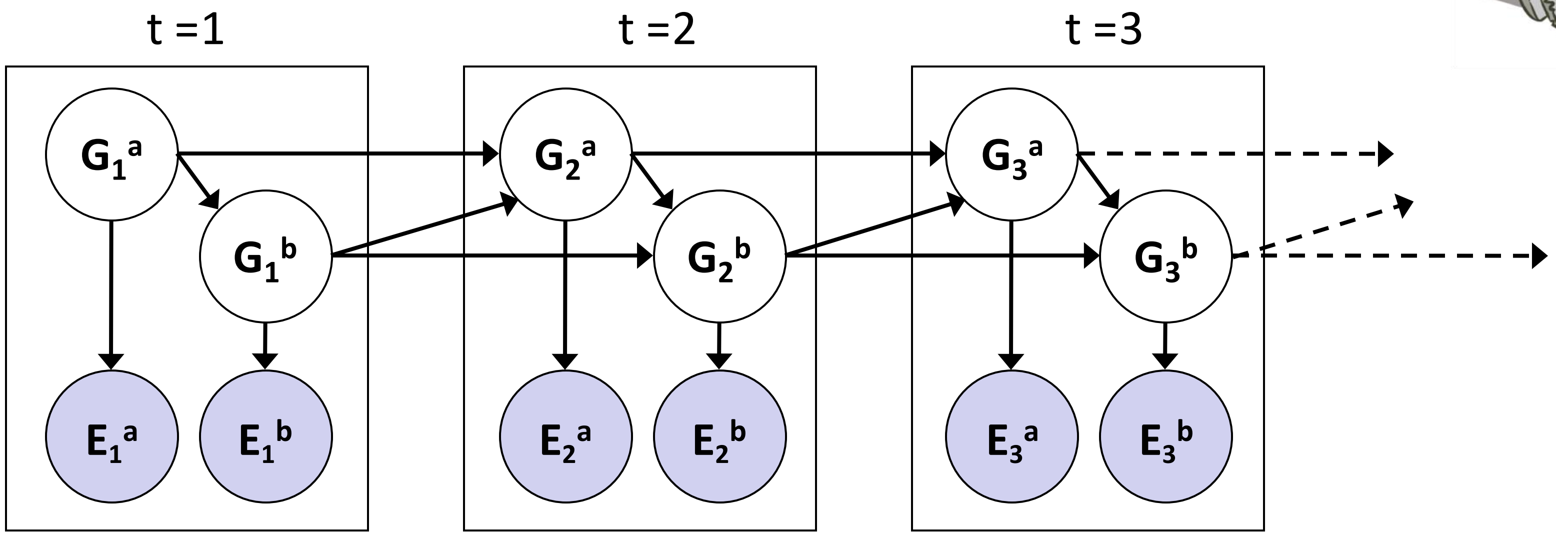
Dynamic Bayes Nets



Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time *t* can condition on those from *t-1*

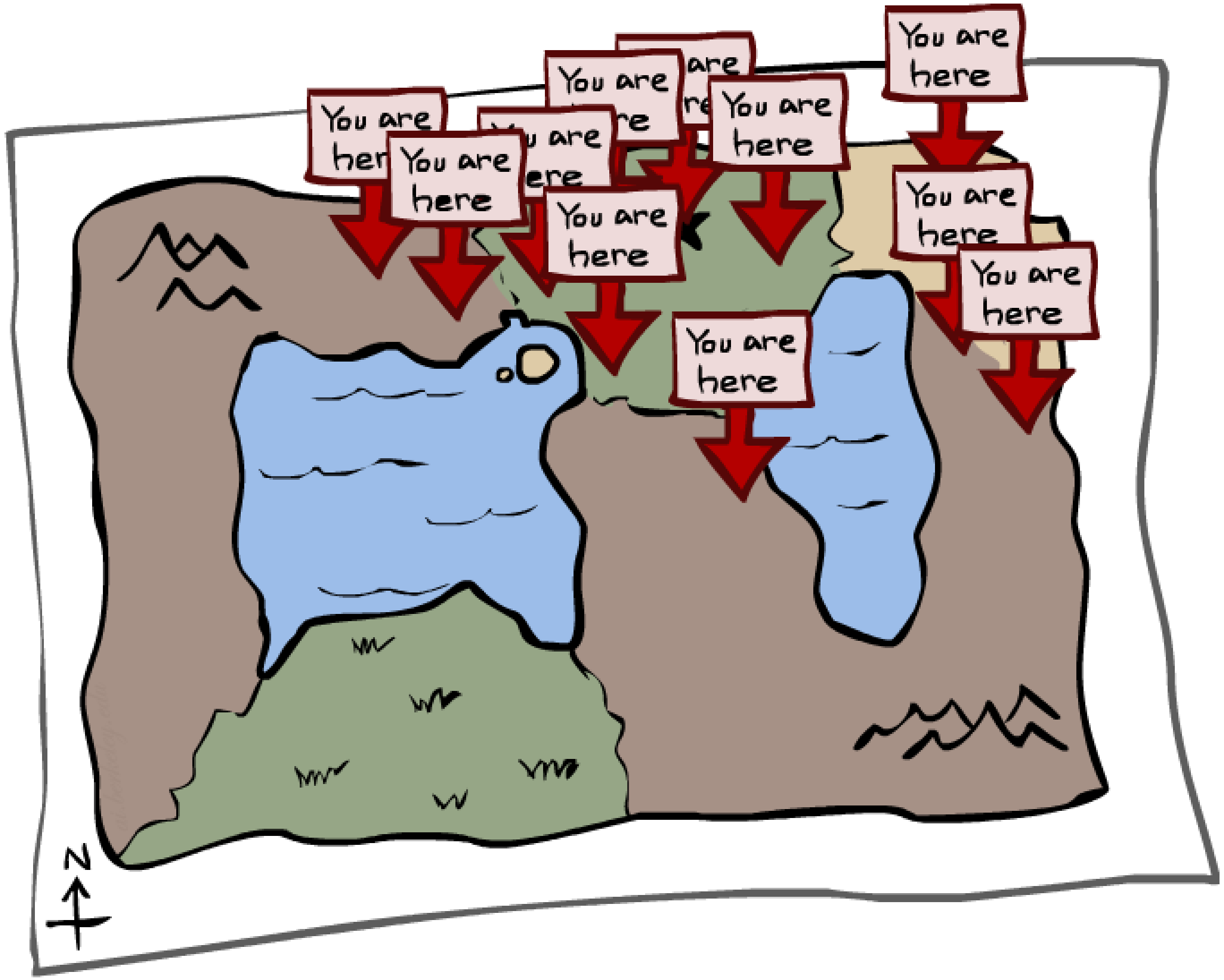




DBNs and HMMs

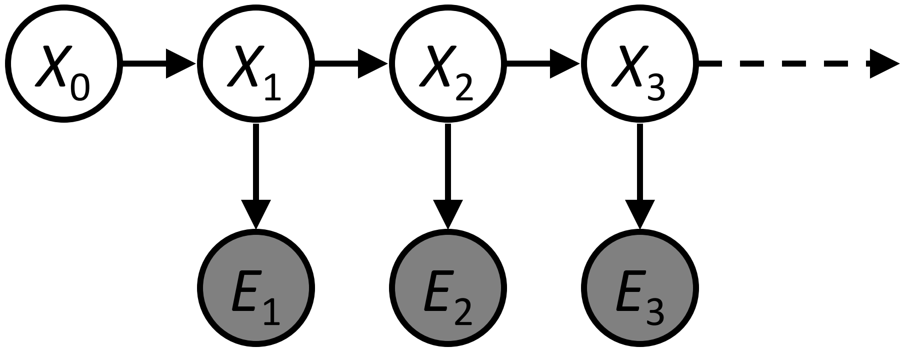
- Every HMM is a DBN
- Every discrete DBN can be represented by a HMM
  - Each HMM state is Cartesian product of DBN state variables
    - E.g., 3 binary state variables => one state variable with 2^3 possible values
  - Advantage of DBN vs. HMM?
    - Sparse dependencies => exponentially fewer parameters
    - E.g., 20 binary state variables, 2 parents each;
      DBN has $20 \times 2^{2+1} = 160$ parameters, HMM has $2^{20} \times 2^{20} =\sim 10^{12}$ parameters

Particle Filtering



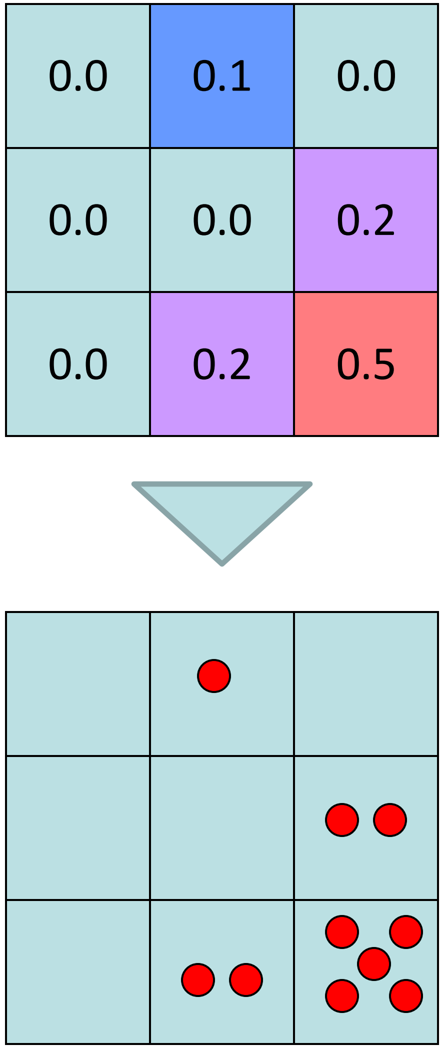
Large state space

- When $|X|$ is huge (e.g., position in a building), exact inference becomes infeasible
- Can we use approximate inference, e.g., likelihood weighting?
  - Evidences are “downstream”
  - By ignoring the evidence: with more states sampled over time, the weight drops quickly (going into low-probability region)
  - Hence: too few “reasonable” samples



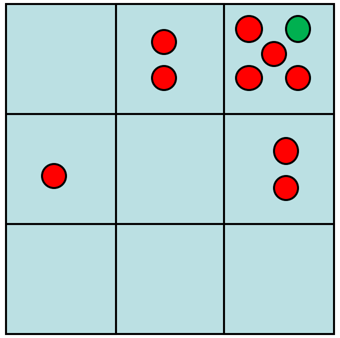
Particle Filtering

- Represent belief state at each step by a set of samples
  - Samples are called ***particles***
- Our representation of P(X) is now a list of N particles (samples)
  - P(x) approximated by number of particles with value x
    - So, many x may have P(x) = 0
  - Generally, $N \ll |X|$
    - More particles, more accuracy; but a large N would defeat the point.



Representation: Particles

- Initialization
  - sample N particles from the initial distribution $P(X_0)$
  - All particles have a weight of 1



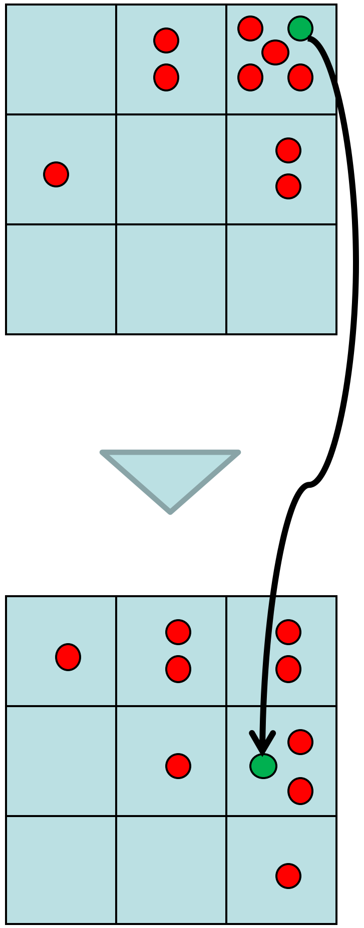
Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)

Particle Filtering: Propagate forward

- Each particle is moved by sampling its next position from the transition model:
  - $x_{t+1} \sim P(X_{t+1} \mid x_t)$

- This captures the passage of time
  - If enough samples, close to exact probabilities (consistent)

Particles:
(3,3)
(2,3)
(3,3)
(3,2)
(3,3)
(3,2)
(1,2)
(3,3)
(3,3)
(2,3)



Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

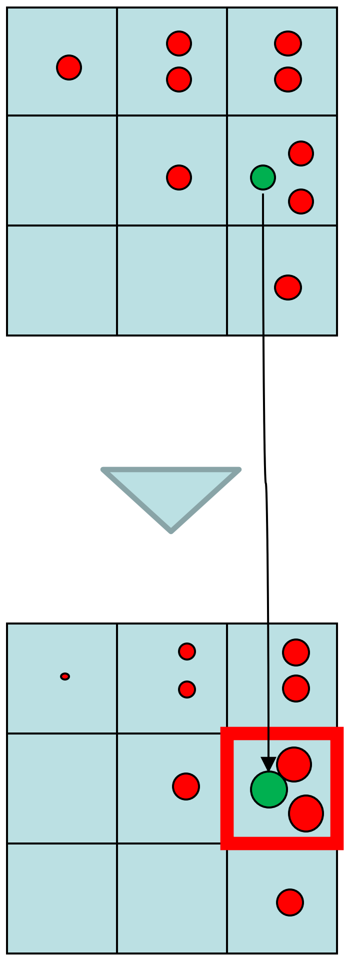
Particle Filtering: Observe

- Similar to likelihood weighting, weight samples based on the evidence
  - $W = P(e_t | x_t)$
  - Particles that fit the evidence better get higher weights, others get lower weights

- What happens if we repeat the Propagate-Observe procedure over time?
  - It is exactly likelihood weighting (if we multiply the weights)
  - Weights drop quickly…

Particles:
(3,2)
(2,3)
(3,2)
(3,1)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(2,2)

Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

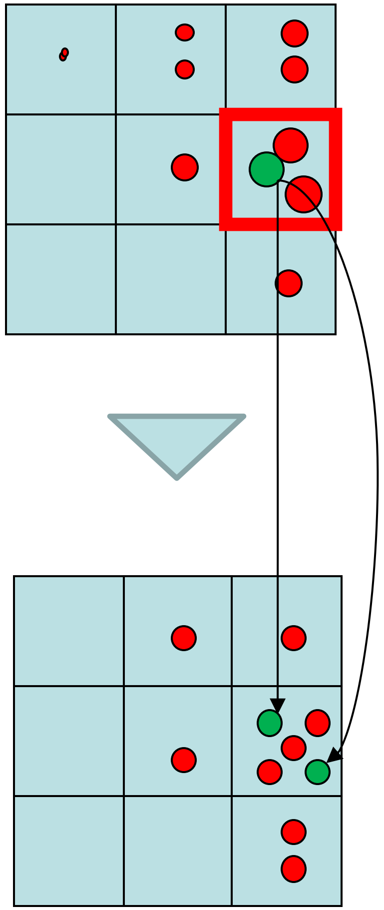


Particle Filtering: Resample

- Rather than tracking weighted samples, we ***resample***
  - Generate *N* new samples from our weighted samples
  - Each new sample is selected from the current population of samples; the probability is proportional to its weight.
  - The new samples have weight of 1
- Now the update is complete for this time step, continue with the next one

Particles:
(3,2) w=.9
(2,3) w=.2
(3,2) w=.9
(3,1) w=.4
(3,3) w=.4
(3,2) w=.9
(1,3) w=.1
(2,3) w=.2
(3,2) w=.9
(2,2) w=.4

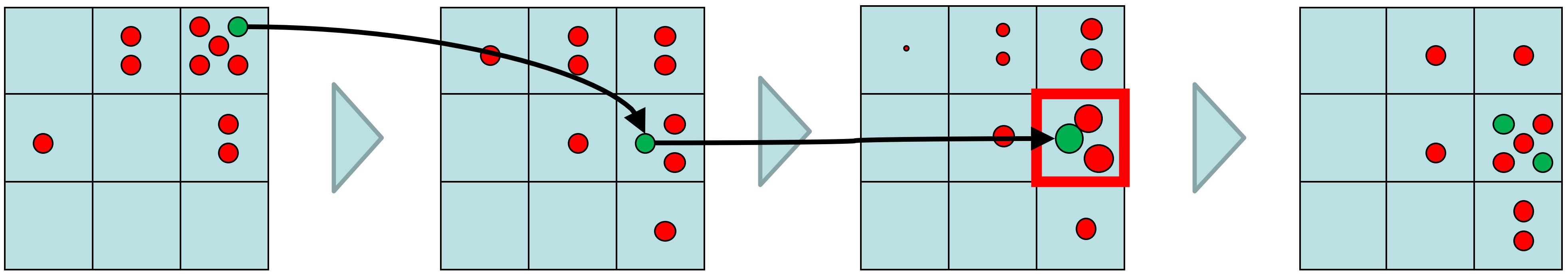
(New) Particles:
(3,2)
(2,2)
(3,2)
(2,3)
(3,3)
(3,2)
(1,3)
(2,3)
(3,2)
(3,2)



Summary: Particle Filtering

- Particles: track samples of states rather than an explicit distribution

Propagate forward | Weight | Resample



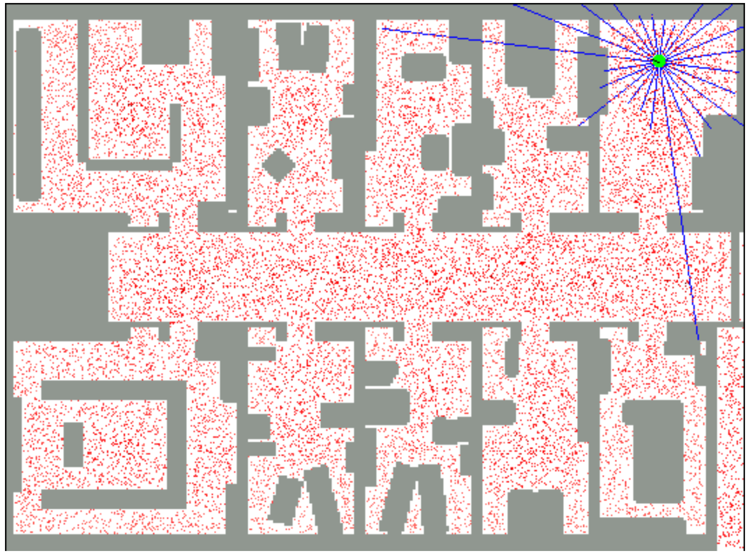
$\approx P(X_t|e_{1:t})$

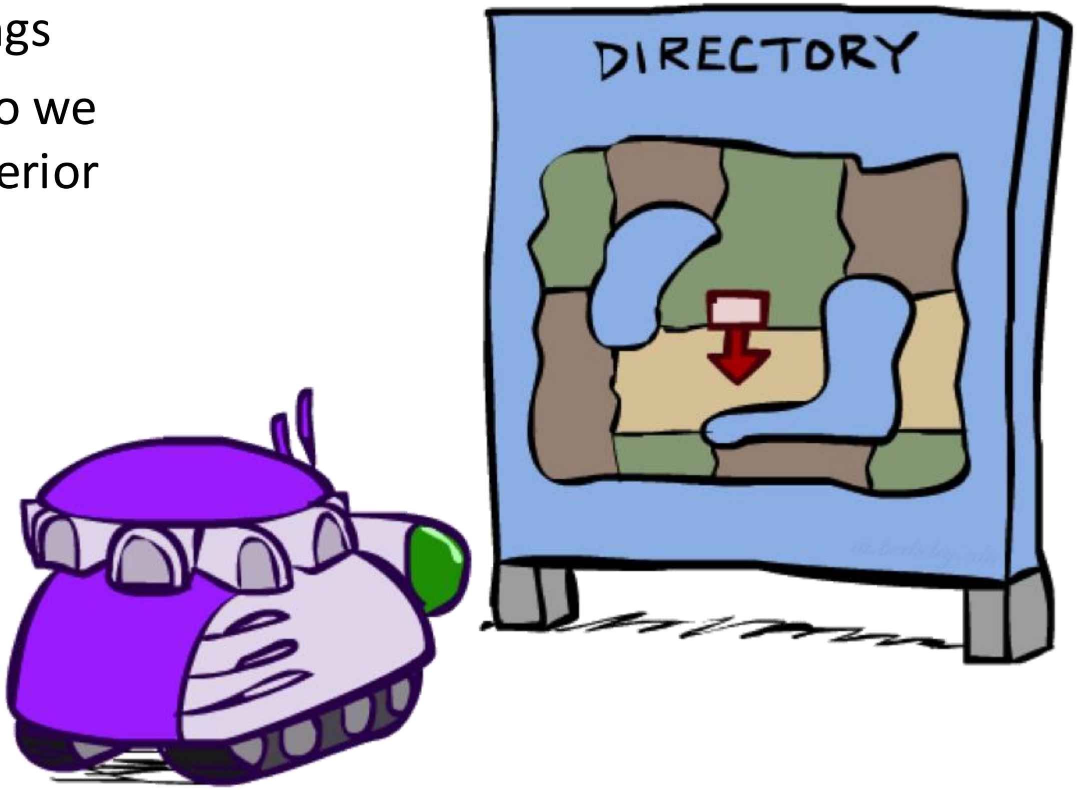
Particles:	Particles:	Particles:	(New) Particles:
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,3)	(2,3) w=.2	(2,2)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(3,2)	(3,1)	(3,1) w=.4	(2,3)
(3,3)	(3,3)	(3,3) w=.4	(3,3)
(3,2)	(3,2)	(3,2) w=.9	(3,2)
(1,2)	(1,3)	(1,3) w=.1	(1,3)
(3,3)	(2,3)	(2,3) w=.2	(2,3)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,2)	(2,2) w=.4	(3,2)

Consistency: see proof in AIMA Ch. 15

Robot Localization

- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous so we cannot usually represent or compute an exact posterior
  - Particle filtering is a main technique

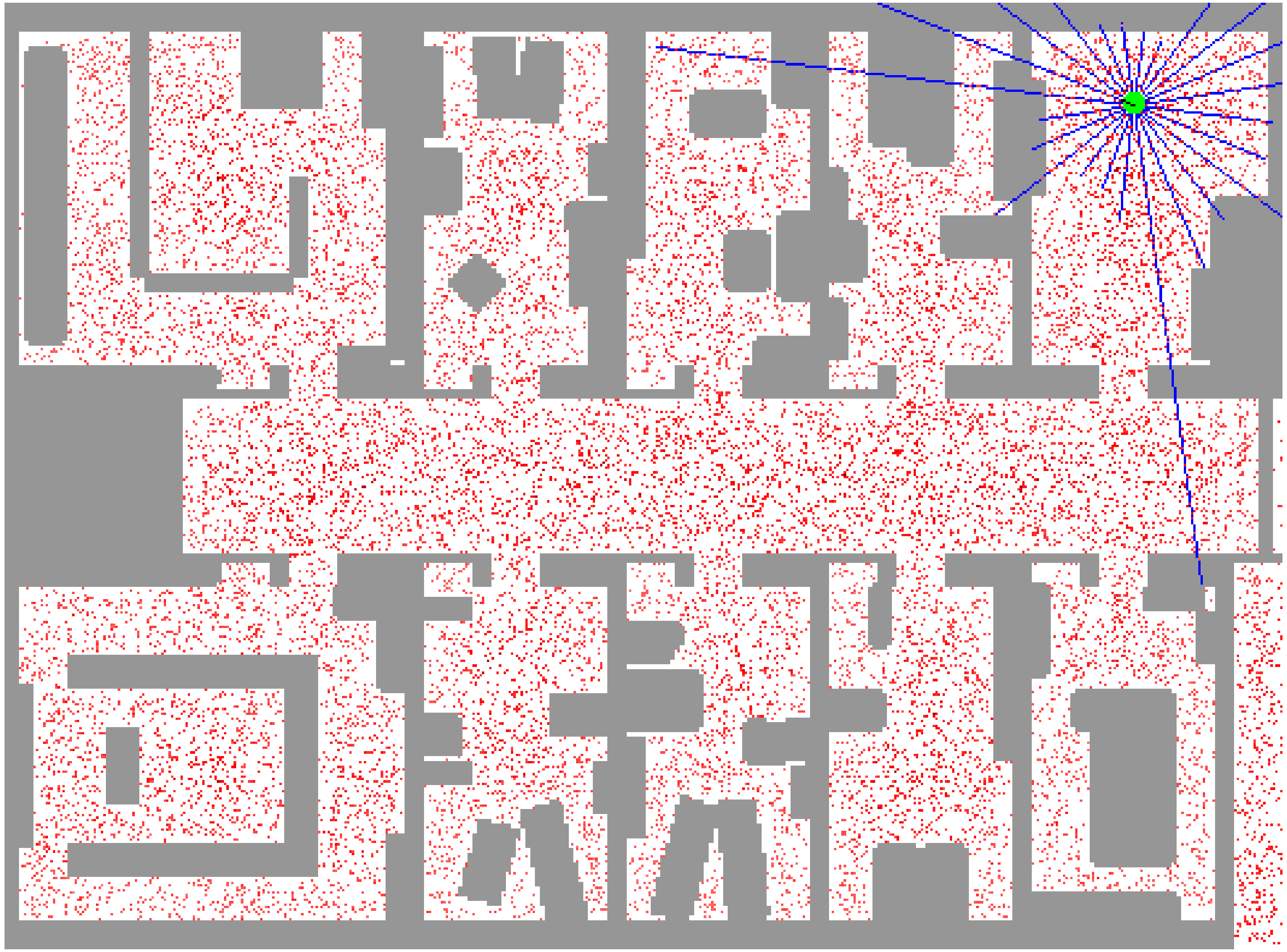




Particle Filter Localization (Sonar)

[Dieter Fox, et al.]

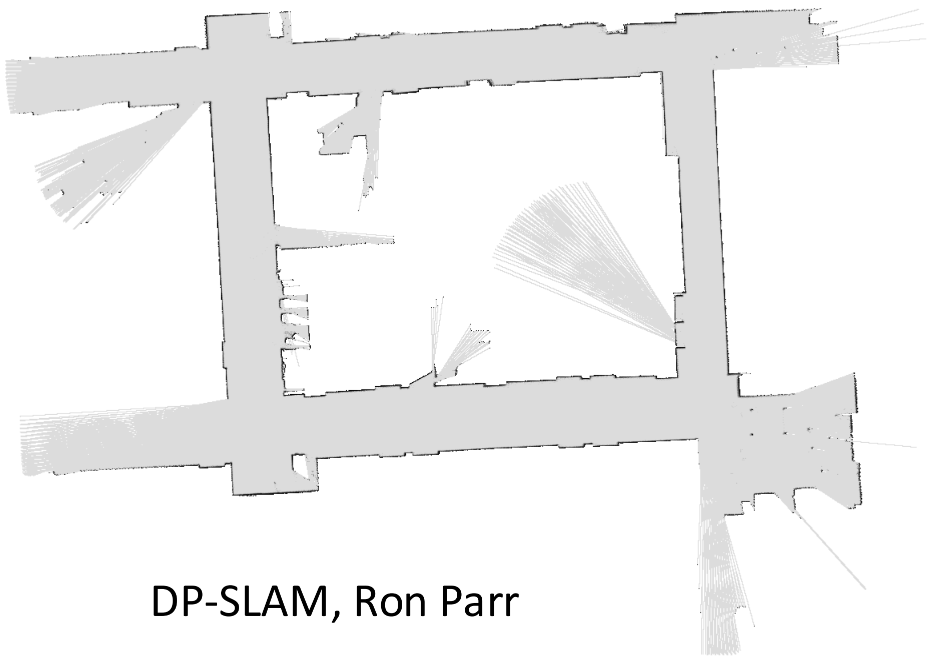
Particle Filter Localization (Laser)



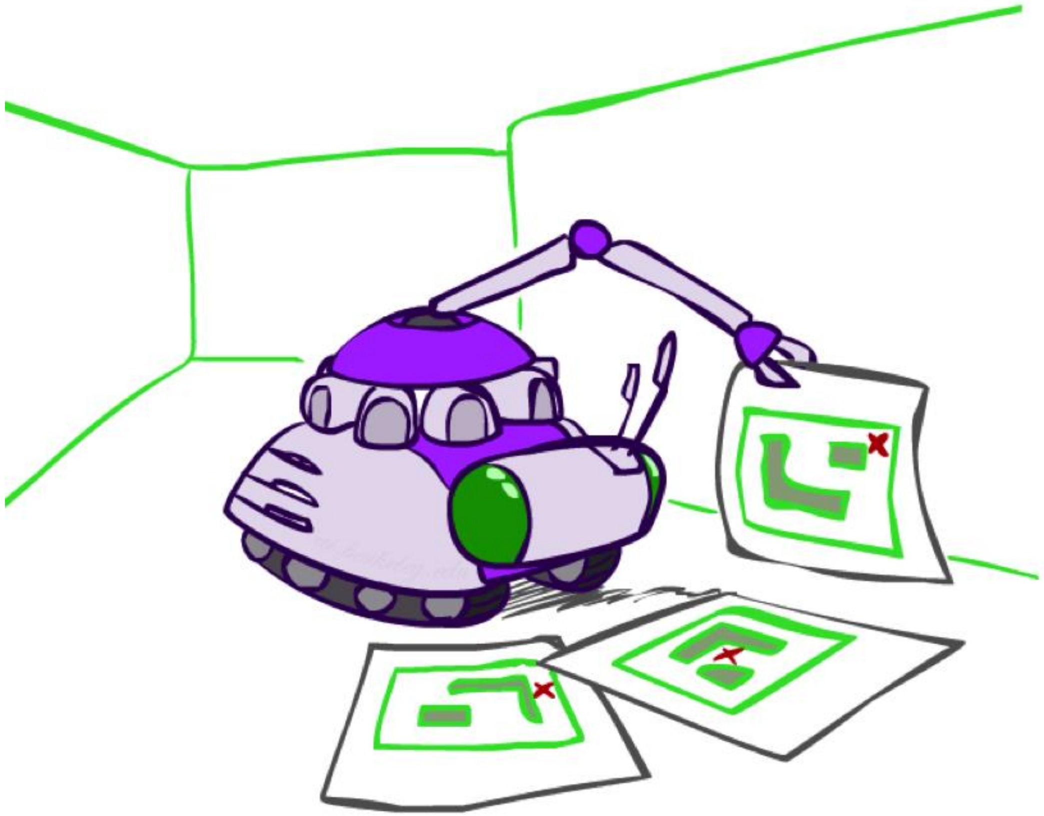
[Dieter Fox, et al.]

Robot Mapping

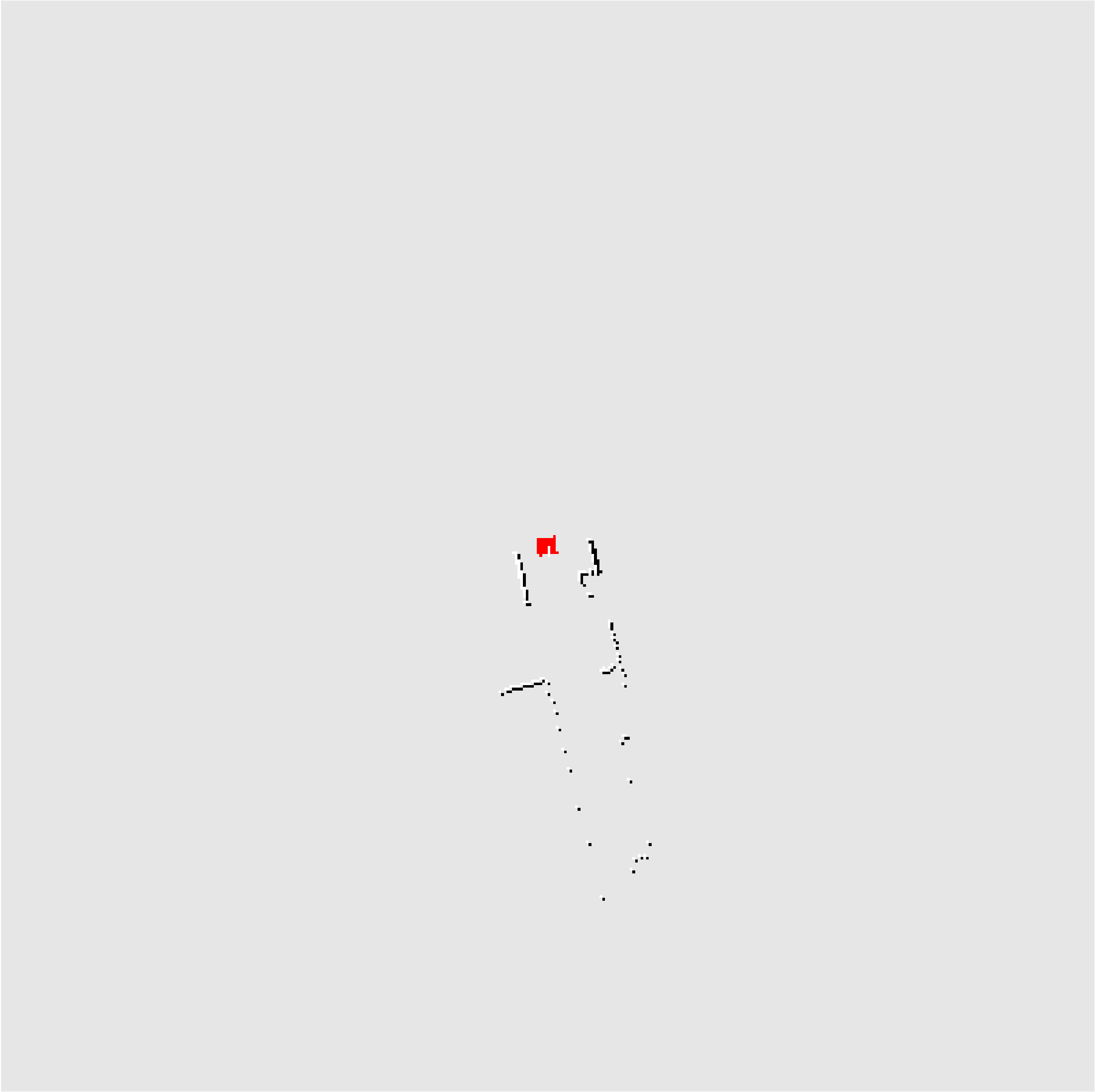
- SLAM: Simultaneous Localization And Mapping
  - We do not know the map or our location
  - State consists of position AND map!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods



DP-SLAM, Ron Parr

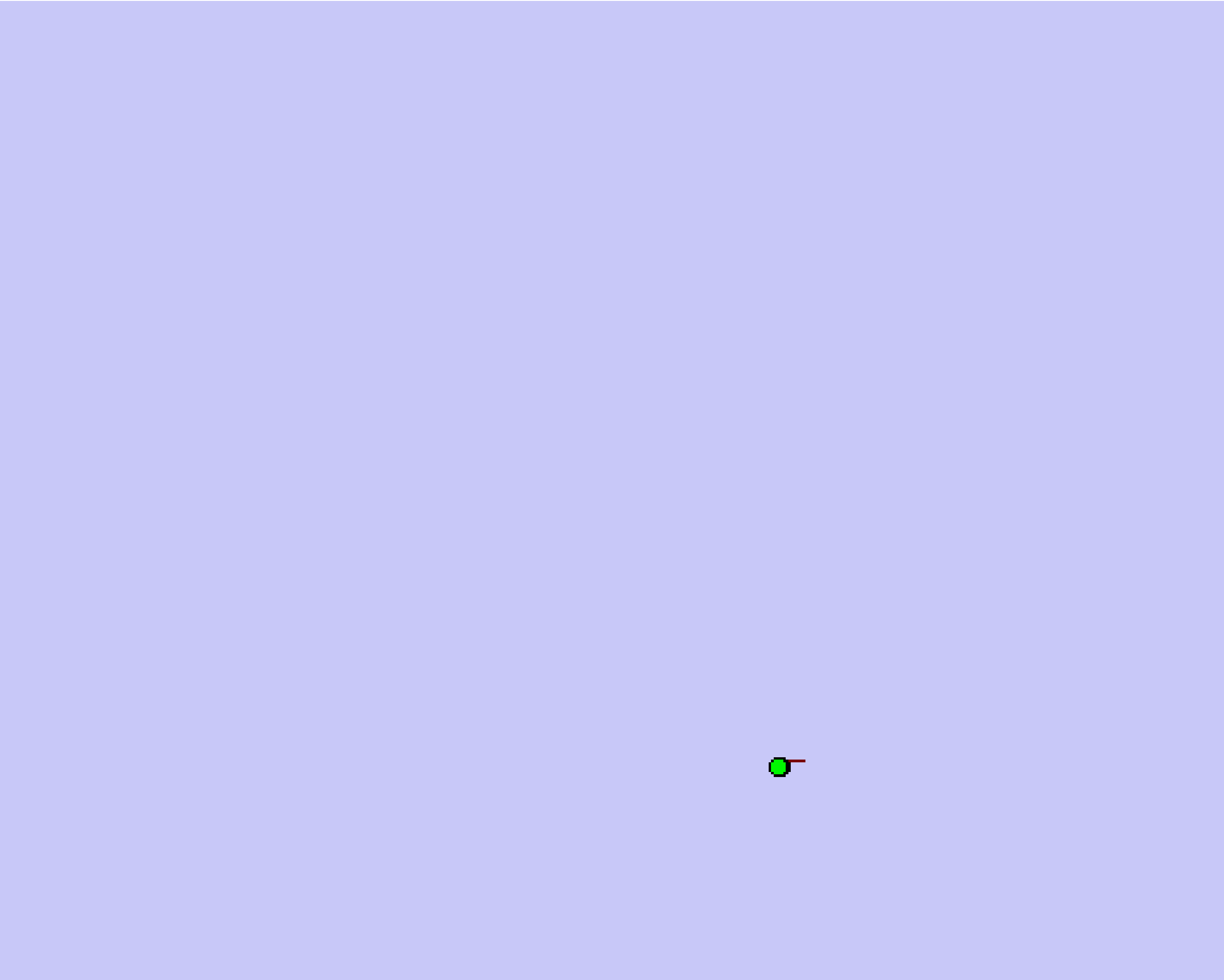


Particle Filter SLAM – Video 1



[Sebastian Thrun, et al.]

Particle Filter SLAM – Video 2



[Dirk Haehnel, et al.]

Summary

- Probabilistic temporal models
  - Markov model
  - Hidden Markov model
    - Filtering: forward algorithm
    - MLE: Viterbi algorithm
  - Dynamic Bayesian network
  - Approximate inference by particle filtering